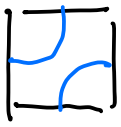
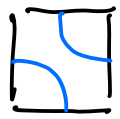
Loop Models & Link Patterns

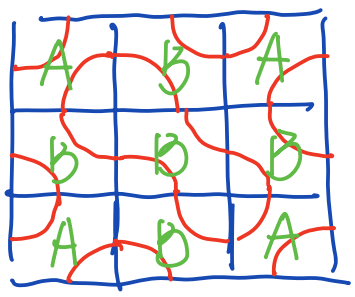
1.] Completely Packed Loops (CPL)

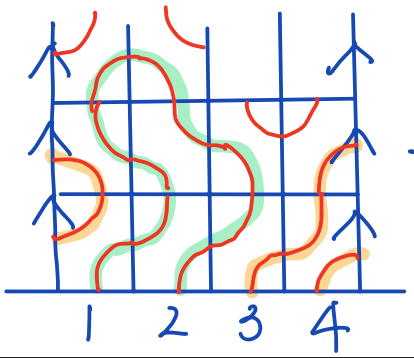
- tiling of a region (on some surface) by 2 plaquettes:

A  B 

Example:

- Plane
- Cylinder

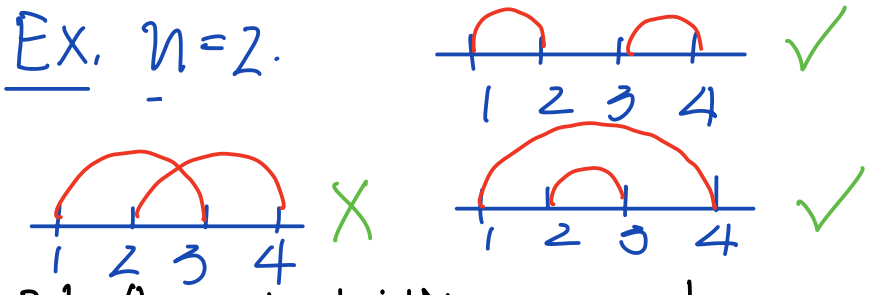




→ gives curves that connects boundary vertices

Def: A Link pattern $P \in L_{2n}$ is a non-crossing matching of $2n$ vertices.

Ex. $n=2$.



2] A probability puzzle.

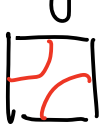
- Sample iid plaquettes A, B with prob : p $1-p$
- Tile them row-by-row on a cylinder.

Q: What is the probability of getting a certain Link patt?

[Simulation here] $\psi_L = (\psi_\pi)_{\pi \in L}$

3] Method: Build a Markov Chain Model

- Each link pattern $\pi \in L_{2n}$ is a state
- $T_{\pi_2,\pi_1}(p) \triangleq$ probability of sample 1 row of plaquettes taking π_1 to π_2
- Assign local weights to plaquettes

 $= u$  $= 1$

where $p \propto u \Rightarrow \boxed{p = \frac{u}{u+1}}$

Assign Global weight τ to each loop.

Compute total weights

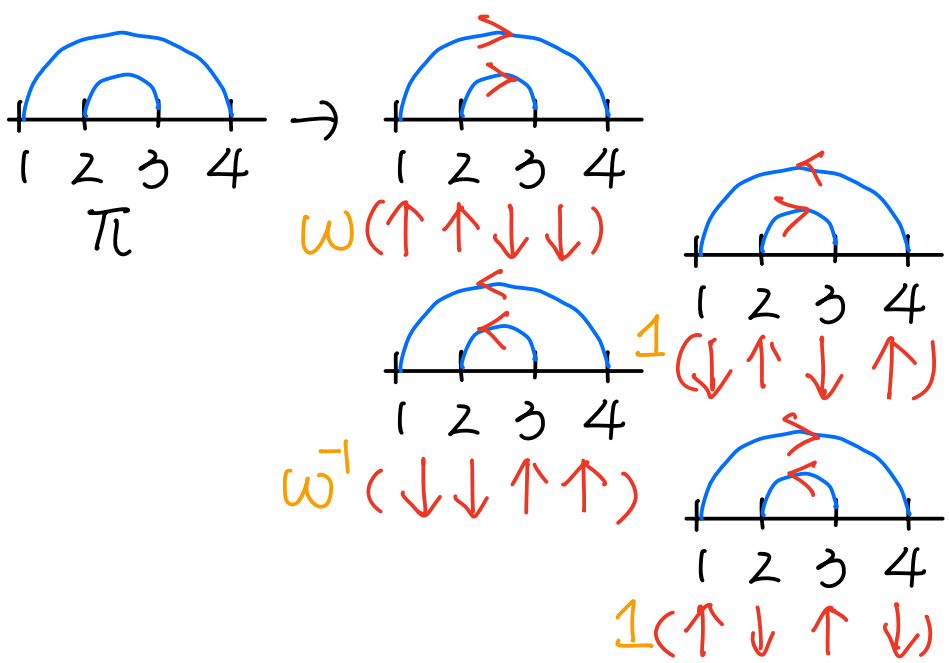
$T_{\pi_2,\pi_1}(u,\tau) \to \begin{cases} \tau = 1 \\ u = p/(1-p). \end{cases}$

Q.] How to compute $T_L(p)$?

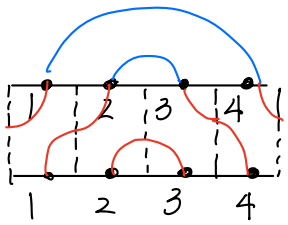
Method: Relate this to Six-Vertex Model's transfer mtx.

Step 1. Link pattern ⇝ vector in $V_1 \otimes \cdots \otimes V_{2L}$

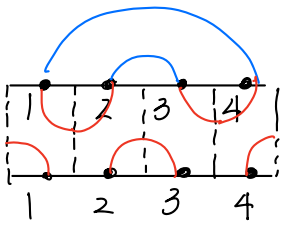
- orient the curves,
- put weight: $w^{\frac{1}{4}}$: ↱ , $w^{-\frac{1}{4}}$: ↰



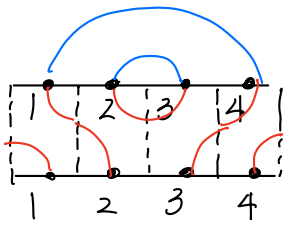
Running Example: $T_{(14)(23),(14,23)}$.



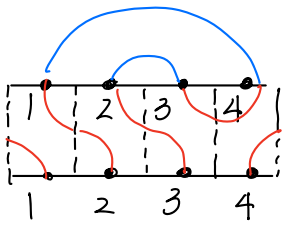
$u^2\cdot\tau$



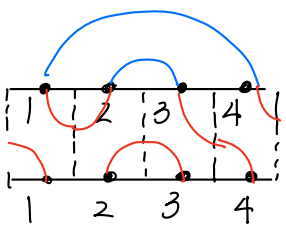
$u^2\tau$



$u^2\tau$



$u\cdot$



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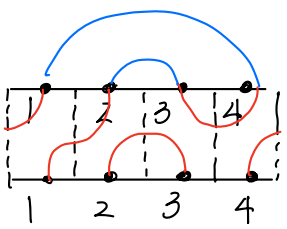
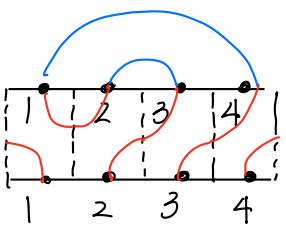


u^3



u^3

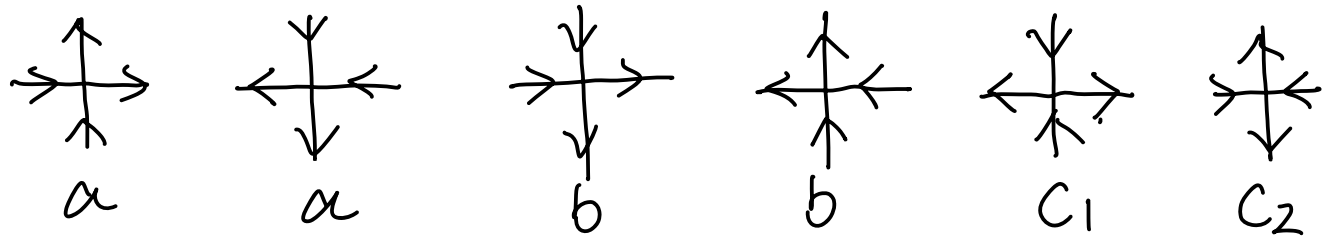
normalize

$$T_{\pi_1,\pi_1}(u,\tau) = 3u^2\tau + 2u + 2u^3 \quad \Big\uparrow = 3p^2(1-p)^2 + 2p(1-p)^3 + 2p^3(1-p).$$

$$T_{\pi_1,\pi_1}\left(\frac{p}{1-p},1\right) = 3\frac{p^2}{(1-p)^2} + 2\frac{p}{1-p} + 2\frac{p^3}{(1-p)^3}$$

Appendix: (Transfer mtx for 6V model)

* 6V:

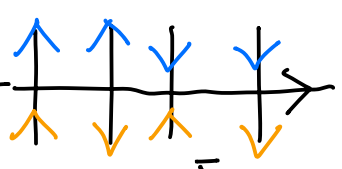


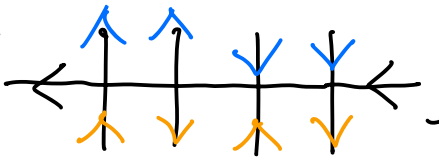
* Transfer mtx for a period strip of length ℓ with a twist Ω

- $V_0 = \mathrm{Span}\{\leftarrow, \rightarrow\}$, $V_i = \mathrm{Span}\{\uparrow, \downarrow\}$ $i = 1, \cdots, \ell$
- $T_\ell : V_1 \otimes V_2 \otimes \cdots \otimes V_\ell \to V_1 \otimes V_2 \otimes \cdots \otimes V_\ell$

  s.t. $T_\ell(\uparrow\downarrow\uparrow\downarrow, \uparrow\uparrow\downarrow\downarrow)$ (end, start) ↙ cumulative weights of intermediate confi

  $= \sum w^{-1}\cdot \mathrm{wt}\left(\rightarrow\right) + \sum w\cdot \mathrm{wt}\left(\leftarrow\right)$.





- For $\begin{cases} a = xq - x^{-1}q^{-1} \\ b = x - x^{-1} \\ c = q - q^{-1} \end{cases}$ $\Longrightarrow [T_\ell(x), T_\ell(x')] = 0$. $\forall x, x'$ Yang-Baxter

$V_\pi = w(\uparrow\uparrow\downarrow\downarrow) + c(\uparrow\downarrow\uparrow\downarrow) + c(\downarrow\uparrow\downarrow\uparrow) + w^{-1}c(\downarrow\downarrow\uparrow\uparrow)$

Fact 1: $\{V_\pi, \pi \in L_{2n}\}$ are linear indep

Let $W \triangleq \text{Span}\{V_\pi, \pi \in L_{2n}\}$

Step 2. Apply 6V Transfer matrix T_ℓ to V_π.

Note: with respect to turning rule

$a = \quad = \quad = u \cdot w^{\frac{1}{4}} w^{-\frac{1}{4}} = u$

$a = \quad = \quad = u$

only 1 way to break apart to curves

$b = \quad = \quad = 1 \quad = \quad = 1$

$c_1 = \quad = \quad + \quad = u w^{\frac{1}{2}} + w^{-\frac{1}{2}}$

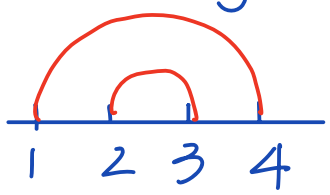
$c_2 = \quad = \quad + \quad = u w^{-\frac{1}{2}} + w^{\frac{1}{2}}$

Fact 2: $T_\ell(W) \subseteq W$.

thus:

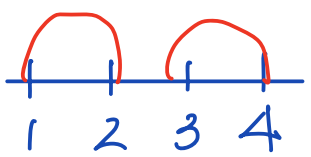
$$T_\ell(V_{\pi_1}) = \sum_{\pi_2 \in L_{2n}} (\overline{T}_{\pi_2, \pi_1}) \cdot V_{\pi_2}$$

Running Example. Compute T_{π_1, π_1} using 6V formulation



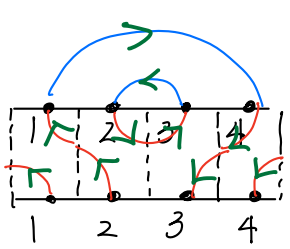
$\rightsquigarrow V_1 = w(\uparrow\uparrow\downarrow\downarrow) + w^{-1}c(\downarrow\downarrow\uparrow\uparrow) + c(\uparrow\downarrow\uparrow\downarrow) + c(\downarrow\uparrow\downarrow\uparrow)$

only present in V_1



$\rightsquigarrow V_2 = w c(\uparrow\downarrow\uparrow\downarrow) + w^{-1}c(\downarrow\uparrow\downarrow\uparrow) + c(\uparrow\downarrow\downarrow\uparrow) + c(\downarrow\uparrow\uparrow\downarrow)$

- Let $T_{2n}(V_1) = \alpha V_1 + \beta V_2$. then $T_{\pi_1, \pi_1} = \alpha = \dfrac{\text{coeff of } (\uparrow\uparrow\downarrow\downarrow) \text{ in } T_L(V_1)}{w}$
- coeff of $(\uparrow\uparrow\downarrow\downarrow)$ in $T_{2n}(V_1)$
  = weighted sum of ways to go from arrows of V_1 to $(\uparrow\uparrow\downarrow\downarrow)$ in 2 periodic conditions $\longrightarrow$ & $\longleftarrow$?



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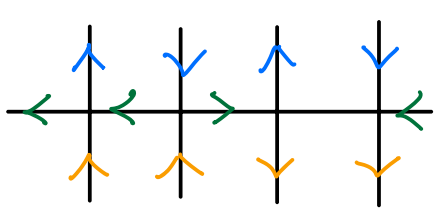
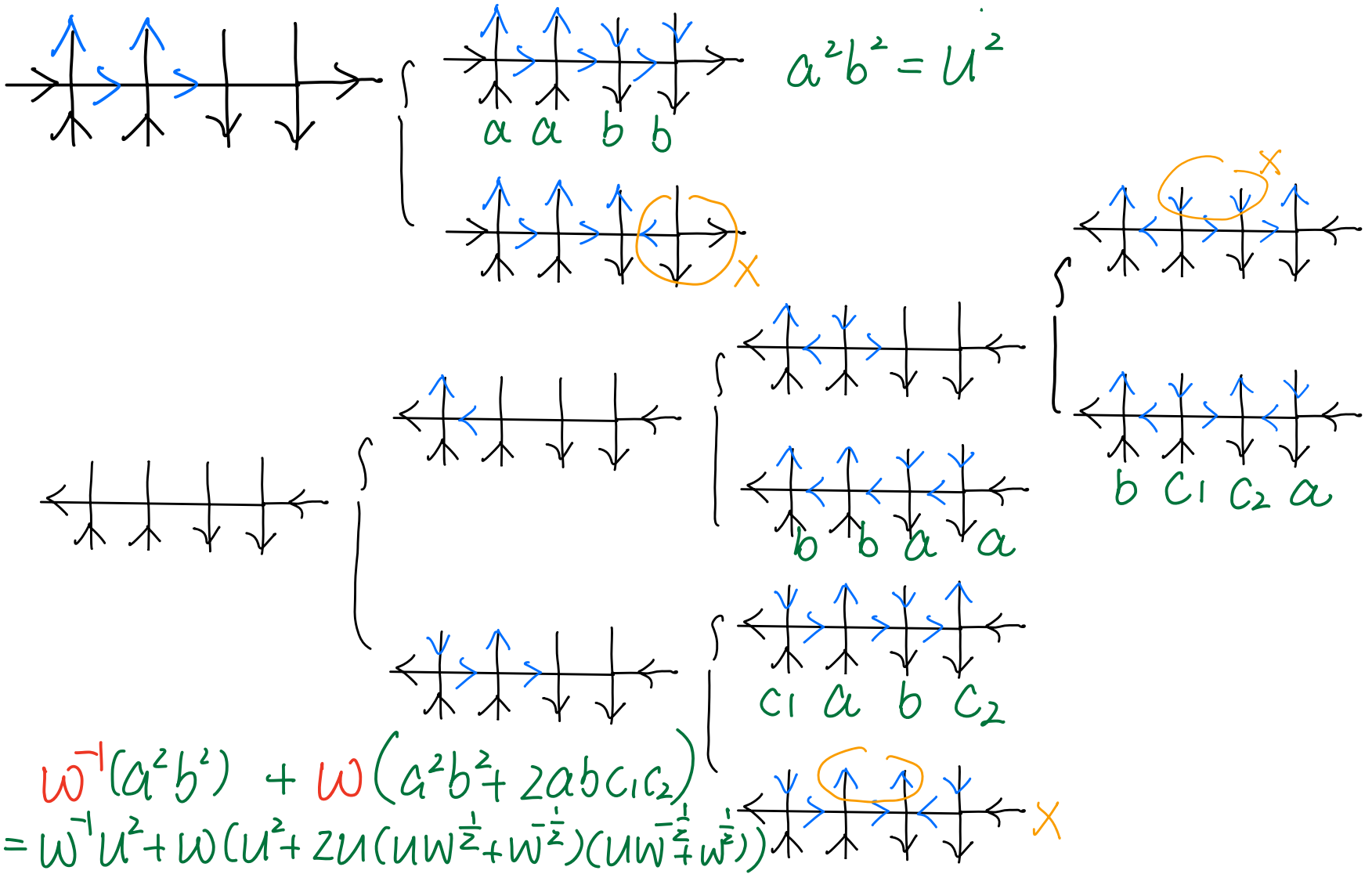




$$w^{-1}(a^2b^2) + w(a^2b^2 + 2abc_1c_2)$$
$$= w^{-1}u^2 + w(u^2 + 2u(uw^{\frac{1}{2}} + w^{-\frac{1}{2}})(uw^{-\frac{1}{2}} + w^{\frac{1}{2}}))$$

4| Coming back to the puzzle:

*Fact3: $T_{2n}^n(p)$ has all entries >0

Recall: $T_{2n}(p) \cdot \psi_{2n} = \psi_{2n}$.

*Thm. (Perron-Frobenius).

- $T_{2n}(p)$ has a unique largest eigenvalue,
- whose eigenspace has dim = 1
- & has a strictly positive eigenvector

*Columns of $T_{2n}(p)$ sum to 1

$T_L\psi = \lambda\psi \Rightarrow \boxed{\lambda = 1}$.

$\Rightarrow \psi \parallel \psi_L$

* $T_{2n}(p)$ commutes with $T_{2n}(p')$.

$T_{2n}(p)T_{2n}(p') = T_{2n}(p')T_{2n}(p)$

$\Rightarrow T_{2n}(p)[T_{2n}(p')\psi(p)]$

$= T_{2n}(p')T_{2n}(p)\psi(p)$

$= [T_{2n}(p')\psi(p)]$.

$\Rightarrow T_{2n}(p')\psi(p)$ is e.v. of 1

$\Rightarrow T_{2n}(p')\psi(p) \parallel \psi(p)$. (positive e.v.)

$\Rightarrow T_{2n}(p')\psi(p) = \psi(p)$

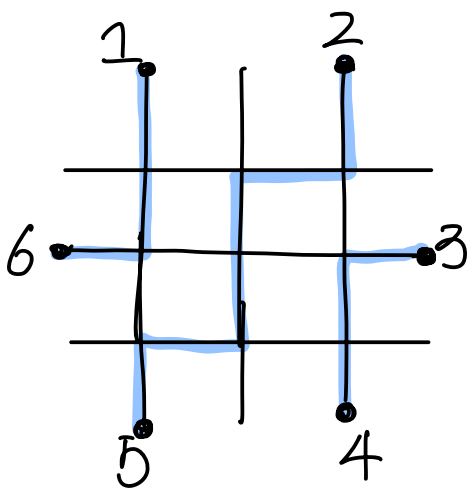
$\Rightarrow \psi(p) = \psi(p')$

Conclusion: this probability ψ does not depend on p.

5] Kamuzov-Stroganov. conj.

$$\Psi_\pi = \frac{A_\pi}{A_n}.$$

A_π = # of Fully Packed Loops with ∂-connectivity = π.



$A_{(16)(2\bar{5})(34)}$

from $(16)(2\bar{5})(34)$ to	p^6	$p^5(1-p)$	$p^4(1-p)^2$	$p^3(1-p)^3$	$p^2(1-p)^4$	$p(1-p)^5$	$(1-p)^6$
any	1	6	15	20	15	6	1
$(23)(56)(14)$			1	2	3	2	1
$(12)(36)(4\bar{5})$	1	2	3	2	1		
$(16)(2\bar{5})(34)$			3	4	3		
$(16)(23)(45)$		1	2	3	2	1	
$(12)(34)(56)$		1	2	3	2	1.	

$$\rightarrow \Psi_{[(16)(2\bar{5})(34)]} = 2\Psi_{[(12)(34)(56)]}.$$