

SIX-VERTEX MODEL II

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ABSTRACT. These notes accompany a talk on the six-vertex model delivered as part of the learning seminar *Solvable Lattice Models in Combinatorics*. The notes may contain errors, and corrections are welcome at junaida@umich.edu. AI assistance was used in preparing the LaTeX source code while the author was preparing for the talk, through an iterative process guided by the author's feedback. The author subsequently worked through and internalized the mathematical content, reviewed and revised the notes, and further refined them with AI assistance until satisfied with the final version.

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1. REVIEW OF THE SIX-VERTEX MODEL

We begin by reviewing the background on the six-vertex model needed for the talk. Consider a square lattice whose edges are oriented. At each vertex we impose the *ice rule*: exactly two arrows point toward the vertex and exactly two point away from it. There are therefore six possible local configurations. We work in the zero-field specialization, so the two vertices of each type have the

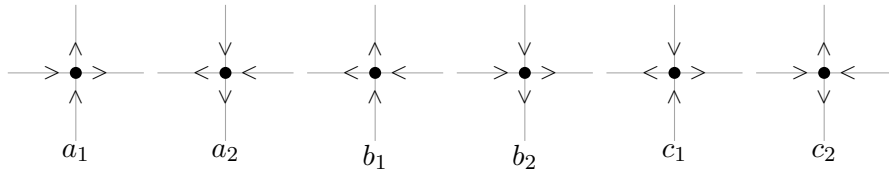


FIGURE 1. The six local arrow configurations satisfying the ice rule.

same weight. Attach a spectral parameter x_i to the i -th horizontal line and y_j to the j -th vertical line. At their intersection, use the weights

$$a_{ij} = q \frac{y_j}{x_i} - q^{-1} \frac{x_i}{y_j}, \quad b_{ij} = \frac{y_j}{x_i} - \frac{x_i}{y_j}, \quad c = q - q^{-1}. \quad (1.1)$$

For these weights,

$$\Delta = \frac{a_{ij}^2 + b_{ij}^2 - c^2}{2a_{ij}b_{ij}} = \frac{q + q^{-1}}{2}. \quad (1.2)$$

The associated R -matrix satisfies the Yang–Baxter equation. We will use its graphical consequence, which allows an auxiliary crossing to be moved through the lattice by successive local Yang–Baxter

moves without changing the weighted partition function for fixed external arrows. The boundary conditions used throughout the second half of the talk are the *domain-wall boundary conditions*, which require the arrows on the left and right boundary edges to point into the square, while those on the top and bottom boundary edges point outward.

2. THE FREE-FERMION POINT AND NONINTERSECTING PATHS

Given a six-vertex arrow configuration, retain precisely the horizontal edges whose arrows point to the right and the vertical edges whose arrows point upward. Regard the retained horizontal edges as directed east and the retained vertical edges as directed north. The ice rule ensures that the retained edges join into north-east going paths. At an a_1 -vertex two paths may meet at the

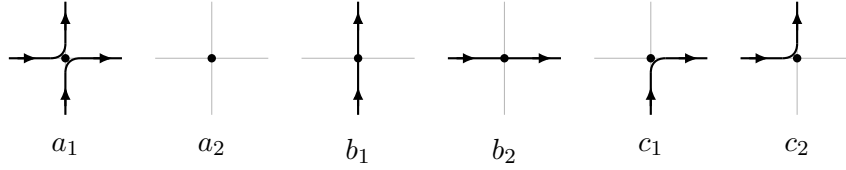


FIGURE 2. The path interpretation of the six local vertex configurations. The black directed paths follow the edges whose arrows point right or upward. At an a_1 -vertex, two paths meet at the vertex but do not share an edge.

same lattice vertex without sharing an edge.

Definition 2.1. A family of directed lattice paths is called *osculating* if distinct paths may meet at vertices but do not share edges.

Under domain-wall boundary conditions, an $n \times n$ configuration produces n north-east going osculating paths entering through the left boundary and leaving through the top boundary. In the zero-field model, $\Delta = 0$ if and only if $a^2 + b^2 = c^2$. For general six-vertex weights, the corresponding relation is

$$a_1 a_2 + b_1 b_2 = c_1 c_2. \quad (2.1)$$

We call Equation (2.1) the *free-fermion relation* and $\Delta = 0$ the *free-fermion point*. At this point, the osculating path model admits a local refinement in which the osculations are resolved into genuinely nonintersecting paths. To describe this refinement, replace each vertex by the directed network in Figure 3. The five directed edges carry weights $\alpha, \beta, \gamma, \delta, \epsilon$. Paths enter from the west or south and leave through the north or east, and no two paths are allowed to use the central vertex simultaneously.

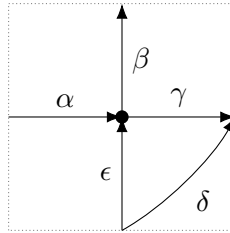


FIGURE 3. The local directed network used to refine the six-vertex path model.

The six possible boundary connectivities of this network reproduce the six local six-vertex path configurations. Their weights are

$$a_1 = \alpha\beta\delta, \quad a_2 = 1, \quad (2.2)$$

$$b_1 = \beta\epsilon, \quad b_2 = \alpha\gamma, \quad (2.3)$$

$$c_1 = \delta + \epsilon\gamma, \quad c_2 = \alpha\beta. \quad (2.4)$$

To see these formulas, consider each local connectivity separately. The empty configuration has weight 1, giving $a_2 = 1$. A path from south to north has weight $\epsilon\beta$, giving $b_1 = \beta\epsilon$, while a path from west to east has weight $\alpha\gamma$, giving $b_2 = \alpha\gamma$. A path from west to north has weight $\alpha\beta$, giving $c_2 = \alpha\beta$. A path from south to east has two possible realizations: it may use the curved edge of weight δ , or it may pass through the central vertex and acquire weight $\epsilon\gamma$. Their weights add, giving $c_1 = \delta + \epsilon\gamma$. Finally, if both paths are present, the west-to-north path must pass through the central vertex, so the south-to-east path is forced to use the curved edge. The resulting weight is $\alpha\beta\delta$, giving $a_1 = \alpha\beta\delta$.

Lemma 2.2. *The weights in Equation (2.2) satisfy the free-fermion relation $a_1a_2 + b_1b_2 = c_1c_2$.*

Proof. Using Equation (2.2), we obtain

$$a_1a_2 + b_1b_2 = \alpha\beta\delta + \beta\epsilon\alpha\gamma = \alpha\beta(\delta + \epsilon\gamma) = c_1c_2. \quad (2.5)$$

This completes the proof. $\square$

Thus, the free-fermion relation is precisely the algebraic compatibility condition that allows the local refinement, replacing the original osculating-path model with a directed graph on which the paths are genuinely vertex-disjoint.

Remark 2.3. The correspondence described above is not one-to-one. The a_1 -vertex has a unique vertex-disjoint resolution, which removes the osculation, whereas a c_1 -vertex has two possible local resolutions in the refined network. These have weights δ and $\epsilon\gamma$, so their total contribution is $c_1 = \delta + \epsilon\gamma$. Thus, a single six-vertex configuration may give rise to several refined nonintersecting path configurations. If C is a fixed six-vertex configuration and $\tilde{C}$ ranges over all refined path configurations obtained from C , then

$$\sum_{\tilde{C} \mapsto C} w(\tilde{C}) = w(C). \quad (2.6)$$

Consequently, after replacing every six-vertex vertex by its local refined network and gluing these networks along their common boundary edge, one recovers the original six-vertex partition function.

3. NONINTERSECTING LATTICE PATHS AND THE LGV LEMMA

The local refinement above replaces the osculating paths of the six-vertex model by genuinely nonintersecting paths. We now develop the determinant formula that makes such path models particularly tractable. Let $G = (V, E)$ be a finite directed acyclic graph. Assign to every directed edge $e \in E$ a weight $w(e)$ in a commutative ring.

Definition 3.1. If $P = (e_1, \dots, e_m)$ is a directed path in G , its weight is $w(P) := \prod_{r=1}^m w(e_r)$. For vertices $A, B \in V$, define the single-path partition function

$$K(A, B) := \sum_{P: A \rightarrow B} w(P), \quad (3.1)$$

where the sum is over all directed paths from A to B .

Remark 3.2. Since G is finite and acyclic, there are only finitely many directed paths between any two vertices. Hence, Equation (3.1) is a finite sum.

Now fix starting vertices $A_1, \dots, A_r$ and ending vertices $B_1, \dots, B_r$. For $\sigma \in S_r$, let $\mathcal{P}_\sigma$ be the set of families $\mathbf{P} = (P_1, \dots, P_r)$ such that P_i is a directed path from A_i to $B_{\sigma(i)}$. Set $w(\mathbf{P}) := \prod_{i=1}^r w(P_i)$. We call $\mathbf{P}$ *nonintersecting* if no two paths share a vertex. The determinant of the single-path partition functions is a signed generating function for nonintersecting path families, with the sign determined by the permutation matching the starting and ending vertices. This is Lindström–Gessel–Viennot lemma, which is stated in [Lemma 3.3](#).

Lemma 3.3 (Lindström–Gessel–Viennot lemma). *We have*

$$\det (K(A_i, B_j))_{1 \leq i, j \leq r} = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \sum_{\substack{\mathbf{P} \in \mathcal{P}_\sigma \\ \mathbf{P} \text{ nonintersecting}}} w(\mathbf{P}). \quad (3.2)$$

Proof. Expanding the determinant gives

$$\det (K(A_i, B_j))_{1 \leq i, j \leq r} = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \prod_{i=1}^r K(A_i, B_{\sigma(i)}) = \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) \sum_{\mathbf{P} \in \mathcal{P}_\sigma} w(\mathbf{P}). \quad (3.3)$$

Thus, the determinant is a signed sum over all path families, including intersecting ones. We show that the intersecting families cancel in pairs. Choose a total order on the vertices compatible with the orientation of G . For an intersecting family $\mathbf{P}$, choose the first vertex v in this order that lies on at least two paths, and choose the lexicographically first pair P_i, P_j , with $i < j$, meeting there. Decompose

$$P_i = P_i^{\text{in}} P_i^{\text{out}}, \quad P_j = P_j^{\text{in}} P_j^{\text{out}} \quad (3.4)$$

at v , and exchange the two outgoing portions:

$$P'_i = P_i^{\text{in}} P_j^{\text{out}}, \quad P'_j = P_j^{\text{in}} P_i^{\text{out}}. \quad (3.5)$$

All other paths are unchanged. The new family uses exactly the same weighted edges, so its weight

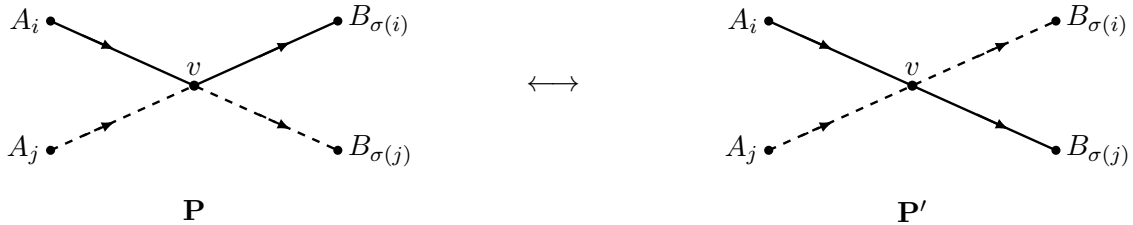


FIGURE 4. The sign-reversing involution in the proof of the LGV lemma. The two paths meet at v , and their outgoing portions are exchanged. The same directed edges occur before and after the exchange, so the weight is unchanged, while the two endpoints are transposed.

is unchanged. Its endpoint permutation differs from the original one by the transposition (ij) , so its sign is reversed. Moreover, this operation is an involution. Indeed, the initial portions of the paths up to v are unchanged, and every vertex appearing after v along either path comes later in the chosen ordering. Consequently, v remains the first intersection vertex, and the same pair of paths is selected. Exchanging their outgoing portions again recovers the original family. Hence, this is a sign-reversing involution on the intersecting families, and all of their contributions cancel. The only terms that remain are the nonintersecting families. $\square$

In the planar situations that occur below, the boundary geometry usually forces the endpoints to be matched in their natural order. This is made precise in [Corollary 3.4](#).

Corollary 3.4. *Suppose the starting and ending vertices are arranged so that every nonintersecting family necessarily connects A_i to B_i for all i . Then*

$$\sum_{\substack{P_i: A_i \rightarrow B_i \\ P_1, \dots, P_r \text{ nonintersecting}}} \prod_{i=1}^r w(P_i) = \det (K(A_i, B_j))_{1 \leq i, j \leq r}. \quad (3.6)$$

Proof. Under the stated compatibility condition, no nonintersecting family can correspond to a nonidentity permutation. Thus, only $\sigma = \text{Id}$ survives in [Lemma 3.3](#). $\square$

Remark 3.5. Informally, [Lemma 3.3](#) and [Corollary 3.4](#) show that the partition function of a system of nonintersecting paths can be expressed as a determinant of single-path partition functions. This is the combinatorial analogue of the free-fermionic principle that many-particle amplitudes are given by determinants of single-particle amplitudes.

Example 3.6. On the ordinary north-east directed square lattice with all edge weights equal to 1, a path from $(0, 0)$ to (r, s) consists of r east steps and s north steps. Hence,

$$K((0, 0), (r, s)) = \binom{r+s}{r}. \quad (3.7)$$

[Lemma 3.3](#) therefore converts suitable determinants of binomial coefficients into counts of nonintersecting path families. To illustrate, let the starting and ending vertices be

$$A_1 = (0, 0), \quad A_2 = (0, 1), \quad B_1 = (2, 1), \quad B_2 = (2, 2). \quad (3.8)$$

We wish to count pairs of nonintersecting paths $P_1 : A_1 \rightarrow B_1$ and $P_2 : A_2 \rightarrow B_2$. Using the binomial-coefficient formula above, the matrix of single-path partition functions is

$$(K(A_i, B_j))_{1 \leq i, j \leq 2} = \begin{pmatrix} \binom{3}{1} & \binom{4}{2} \\ \binom{2}{0} & \binom{3}{1} \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix}. \quad (3.9)$$

Since nonintersecting north-east paths cannot reverse the vertical order of their endpoints, the hypothesis of [Corollary 3.4](#) is satisfied. Hence, the number of nonintersecting pairs is

$$\det \begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix} = 3 \cdot 3 - 6 \cdot 1 = 3. \quad (3.10)$$

Indeed, writing E and N for east and north steps, respectively, the three nonintersecting pairs are

$$(P_1, P_2) = (EEN, ENE), \quad (EEN, NEE), \quad (ENE, NEE). \quad (3.11)$$

Thus, the determinant counts precisely the path families satisfying the nonintersection condition.

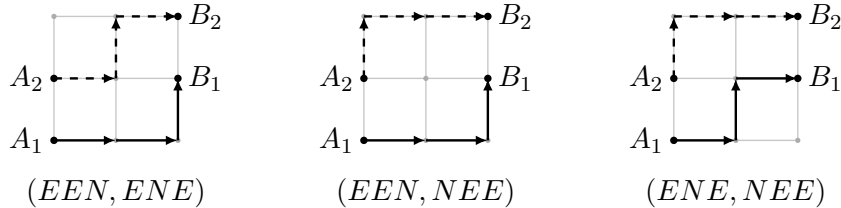


FIGURE 5. The three nonintersecting path families in [Example 3.6](#). The solid path is P_1 and the dashed path is P_2 .

We can now apply [Corollary 3.4](#) to the refined six-vertex paths constructed in the refined path construction above.

Proposition 3.7. *Suppose the six-vertex weights satisfy the free-fermion relation [Equation \(2.1\)](#), and suppose the boundary conditions become ordered starting vertices $A_1, \dots, A_r$ and ending vertices $B_1, \dots, B_r$ after the local refinement. Then the six-vertex partition function is*

$$Z = \det (K(A_i, B_j))_{1 \leq i, j \leq r}, \quad (3.12)$$

up to any fixed normalization introduced by a gauge choice of the local weights.

Proof. The local refinement replaces every six-vertex configuration by one or more vertex-disjoint path configurations, and the sum of the refined weights over the local resolutions equals the original six-vertex weight. Hence, after patching the local networks, the six-vertex partition function is the partition function of a family of nonintersecting directed paths. The planar ordering of the boundary points forces the i -th starting point to connect to the i -th ending point. The result therefore follows from [Corollary 3.4](#). $\square$

Remark 3.8. The non-intersecting path configurations in the refined path construction above can also be reinterpreted as domino tilings of an Aztec diamond. In particular, suitable free-fermion six-vertex weights correspond to weighted domino tilings, while the specialization $a = b = 1$, $c = \sqrt{2}$ corresponds, up to a fixed normalization, to the ordinary Aztec-diamond model. Thus, the free-fermion six-vertex model, nonintersecting lattice paths, and domino tilings are closely related combinatorial models whose partition functions admit determinant formulas. We will not develop this correspondence here.

4. KOREPIN'S CHARACTERIZATION AND THE IZERGIN DETERMINANT

We now return to the six-vertex model with domain-wall boundary conditions and generic integrable weights. Our goal is to understand its partition function explicitly. Consider an $n \times n$ square with domain-wall boundary conditions. Attach spectral parameters $x_1, \dots, x_n$ to the horizontal lines and $y_1, \dots, y_n$ to the vertical lines. Thus, the vertex in horizontal line i and vertical line j has spectral ratio y_j/x_i , and its zero-field weights are

$$a_{ij} = q \frac{y_j}{x_i} - q^{-1} \frac{x_i}{y_j}, \quad b_{ij} = \frac{y_j}{x_i} - \frac{x_i}{y_j}, \quad c = q - q^{-1}. \quad (4.1)$$

Definition 4.1. The domain-wall partition function is

$$Z_n(\mathbf{x}; \mathbf{y}) := \sum_C \prod_{i,j=1}^n w_C(i, j), \quad (4.2)$$

where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, and the sum is over all six-vertex configurations satisfying domain-wall boundary conditions. At the vertex (i, j) , the local weight $w_C(i, j)$ is one of a_{ij}, b_{ij}, c , according to the local configuration.

At first sight, Z_n is a sum over a large number of configurations. However, the Yang–Baxter equation imposes sufficient structure to characterize it recursively. The relevant properties, first identified by Korepin, are stated and proved in [Proposition 4.2](#).

Proposition 4.2 (Korepin's Characterization). *The domain-wall partition functions Z_n satisfy the following properties.*

- (1) (Normalization) $Z_1(x_1; y_1) = c = q - q^{-1}$.
- (2) (Symmetry) $Z_n(\mathbf{x}; \mathbf{y})$ is symmetric in $x_1, \dots, x_n$ and separately symmetric in $y_1, \dots, y_n$.
- (3) (Degree Bound) For each i , $x_i^{n-1} Z_n(\mathbf{x}; \mathbf{y})$ is a polynomial in x_i^2 of degree at most $n - 1$. Similarly, $y_i^{n-1} Z_n(\mathbf{x}; \mathbf{y})$ is a polynomial in y_i^2 of degree at most $n - 1$.

(4) (*Freezing*) If $y_1 = x_1$, then

$$Z_n(x_1, \dots, x_n; x_1, y_2, \dots, y_n) \quad (4.3)$$

$$= c \prod_{i=2}^n \left(q \frac{x_1}{x_i} - q^{-1} \frac{x_i}{x_1} \right) \prod_{j=2}^n \left(q \frac{y_j}{x_1} - q^{-1} \frac{x_1}{y_j} \right) Z_{n-1}(x_2, \dots, x_n; y_2, \dots, y_n). \quad (4.4)$$

Proof. The proof is given below:

- (1) For $n = 1$, the domain-wall boundary arrows force the unique vertex to be of c -type, so $Z_1 = c$.
- (2) It is enough to show that Z_n is unchanged when two adjacent vertical lines with spectral parameters y_j and y_{j+1} are exchanged. Consider the corresponding two vertical lines and insert an auxiliary R -vertex between them just above the lattice. The ice rule forces the state of this auxiliary vertex with weight

$$A(y_j, y_{j+1}) := q \frac{y_{j+1}}{y_j} - q^{-1} \frac{y_j}{y_{j+1}}. \quad (4.5)$$

Thus, the partition function of the enlarged diagram is

$$A(y_j, y_{j+1}) Z_n(\dots, y_j, y_{j+1}, \dots). \quad (4.6)$$

We now move the auxiliary crossing downward through the lattice. At each horizontal line, the auxiliary crossing and the two ordinary six-vertex vertices on that horizontal line form precisely the three-vertex configuration appearing in the Yang–Baxter equation. One application of the Yang–Baxter equation moves the crossing past that horizontal line without changing the partition function of the enlarged diagram. Repeating this move transports the crossing through all n horizontal lines, as illustrated in Figure 6. At the

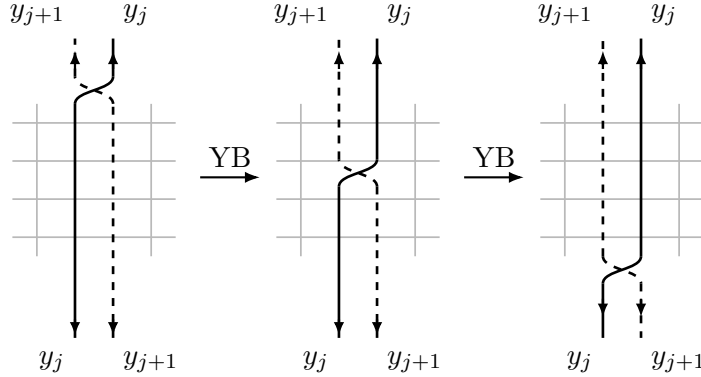


FIGURE 6. The Yang–Baxter train argument.

bottom boundary the two vertical arrows again have the same fixed orientation, so the auxiliary vertex is forced and contributes the same factor $A(y_j, y_{j+1})$. Since every Yang–Baxter move preserves the partition function of the enlarged diagram, we obtain

$$A(y_j, y_{j+1}) Z_n(\dots, y_j, y_{j+1}, \dots) = A(y_j, y_{j+1}) Z_n(\dots, y_{j+1}, y_j, \dots). \quad (4.7)$$

For generic parameters¹ $A(y_j, y_{j+1}) \neq 0$, and therefore

$$Z_n(\dots, y_j, y_{j+1}, \dots) = Z_n(\dots, y_{j+1}, y_j, \dots). \quad (4.8)$$

Thus, Z_n is invariant under every adjacent transposition of the y -variables. Since adjacent transpositions generate the symmetric group, Z_n is symmetric in $y_1, \dots, y_n$. The same argument, with horizontal and vertical lines interchanged, proves symmetry in $x_1, \dots, x_n$.

¹Since both sides are Laurent polynomials in the spectral parameters, the identity extends from generic parameters to the exceptional values as well.

- (3) Fix some x_i . It occurs only in the n vertices of the i -th row. Under domain-wall boundary conditions, the horizontal arrow at the left boundary points to the right and the horizontal arrow at the right boundary points to the left. The horizontal arrow state changes precisely at c -type vertices. Hence, the number of c -type vertices in the row, N_c , is odd. In particular, $N_c \geq 1$. The number of vertices with weight a or b is $r := n - N_c$. We have

$$r \equiv n - 1 \pmod{2}. \quad (4.9)$$

The product of the r parameter-dependent weights has the form

$$\prod_{s=1}^r (\alpha_s x_i + \beta_s x_i^{-1}) = \sum_{k=0}^r c_k x_i^{r-2k}, \quad (4.10)$$

for coefficients c_k independent of x_i . Thus, the possible exponents of x_i are $r, r-2, \dots, -r$. Since $r \equiv n-1 \pmod{2}$, multiplying by x_i^{n-1} gives exponents

$$n-1+r-2k, \quad 0 \leq k \leq r, \quad (4.11)$$

all of which are even. Moreover, since $r \leq n-1$,

$$0 \leq n-1-r \leq n-1+r-2k \leq n-1+r \leq 2(n-1). \quad (4.12)$$

Therefore, after multiplication by x_i^{n-1} , every contribution is a polynomial in x_i^2 of degree at most $n-1$. Summing over all configurations shows that $x_i^{n-1} Z_n$ has the same property. The argument for y_j is identical.

- (4) Set $y_1 = x_1$. We have

$$b_{11} = \frac{y_1}{x_1} - \frac{x_1}{y_1} = 0. \quad (4.13)$$

Thus, every configuration in which that vertex is of b -type has zero weight. With the domain-wall arrows on the left and bottom boundaries, the only nonzero possibility at this distinguished vertex is the c_2 -type configuration. Once this vertex is fixed, the ice rule forces the arrow at the next vertex along the x_1 -row and the next vertex along the y_1 -column. Repeating this argument freezes the entire row and column. See Figure 7 for an illustration. Removing the frozen row and column leaves an $(n-1) \times (n-1)$ domain-wall configuration.

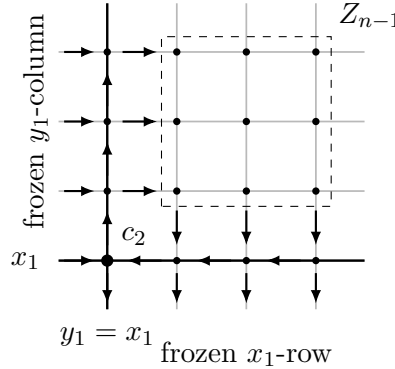


FIGURE 7. Korepin's freezing specialization.

The distinguished vertex contributes c . The forced vertices in the y_1 -column have weights

$$q \frac{x_1}{x_i} - q^{-1} \frac{x_i}{x_1}, \quad i = 2, \dots, n, \quad (4.14)$$

and the forced vertices in the x_1 -row have weights

$$q \frac{y_j}{x_1} - q^{-1} \frac{x_1}{y_j}, \quad j = 2, \dots, n. \quad (4.15)$$

The remaining unfrozen square contributes $Z_{n-1}(x_2, \dots, x_n; y_2, \dots, y_n)$. Multiplying these factors gives Equation (4.3).

This completes the proof. $\square$

The important point is that the properties in Corollary 4.3 are not merely necessary conditions satisfied by the domain-wall partition function. That is, taken together, the normalization, symmetry, degree bound, and freezing recurrence are sufficient to reconstruct Z_n recursively from Z_{n-1} . In particular, they determine Z_n uniquely. This is made precise in Corollary 4.3.

Corollary 4.3. *The properties in Proposition 4.2 uniquely determine the family Z_n .*

Proof. We argue by induction on n . The case $n = 1$ is fixed by $Z_1 = c$. Suppose Z_{n-1} is already known. Regard Z_n as a function of y_1 , with all other parameters fixed. By Proposition 4.2,

$$P(y_1^2) := y_1^{n-1} Z_n \quad (4.16)$$

is a polynomial in y_1^2 of degree at most $n - 1$. The freezing recurrence gives the value of Z_n when $y_1 = x_1$. Since Z_n is symmetric in the x -variables, we may first exchange x_1 with any x_i and then apply the same recurrence. Thus, for each $i = 1, \dots, n$, the recurrence determines Z_n , and hence P , at $y_1^2 = x_i^2$ for $i = 1, \dots, n$. Therefore, P is known at the n points $x_1^2, \dots, x_n^2$. For generic parameters² these points are distinct. A polynomial of degree at most $n - 1$ is uniquely determined by its values at n distinct points, so P , and hence Z_n , is uniquely determined. $\square$

Proposition 4.2 and Corollary 4.3 imply that, to determine the domain-wall partition function, it suffices to find an explicit expression satisfying Korepin's characterization. Izergin obtained such an expression in determinant form. This expression is stated and proved in Proposition 4.4. In what follows, define

$$d(u, v) := \frac{u}{v} - \frac{v}{u} \quad (4.17)$$

and, for the vertex at the intersection of horizontal line i and vertical line j , define

$$F_{ij} := d(y_j, x_i) \left(q \frac{y_j}{x_i} - q^{-1} \frac{x_i}{y_j} \right) = a_{ij} b_{ij}. \quad (4.18)$$

Proposition 4.4 (Izergin Determinant). *The domain-wall partition function is given by the formula*

$$Z_n(\mathbf{x}; \mathbf{y}) = \frac{\prod_{i,j=1}^n F_{ij}}{\prod_{1 \leq i < j \leq n} d(x_i, x_j) d(y_j, y_i)} \det \left(\frac{q - q^{-1}}{F_{ij}} \right)_{1 \leq i, j \leq n}. \quad (4.19)$$

The expression is initially interpreted for generic parameters and extends to coincident parameters after cancellation of the apparent singularities.

Proof sketch. We only indicate the main idea. Exchanging two x -variables exchanges two rows of the determinant and hence changes its sign. The factor $\prod_{1 \leq i < j \leq n} d(x_i, x_j)$ in the denominator also changes sign, since $d(x_i, x_j) = (x_i^2 - x_j^2)/x_i x_j$. The two signs therefore cancel, giving symmetry in the x -variables. The same argument gives symmetry in the y -variables. When $y_1 = x_1$, we have $F_{11} = a_{11} b_{11} = 0$. The product outside the determinant therefore contains a vanishing factor F_{11} , while the $(1, 1)$ -entry of the determinant contains the pole $\frac{c}{F_{11}}$. Expanding the determinant, we have

$$\det \left(\frac{c}{F_{ij}} \right) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \frac{c}{F_{i, \sigma(i)}}. \quad (4.20)$$

²Since all expressions involved are algebraic in the spectral parameters, the identity obtained for generic parameters extends to all values for which the partition function is defined.

If $\sigma(1) \neq 1$, then the corresponding term does not contain $1/F_{11}$, so multiplication by the outside factor F_{11} sends that term to zero. If $\sigma(1) = 1$, then the factor $F_{11} \frac{c}{F_{11}} = c$ survives. Thus, only the terms with $\sigma(1) = 1$ remain, and these are precisely the terms in the determinant minor obtained by deleting row 1 and column 1:

$$F_{11} \det \left(\frac{c}{F_{ij}} \right)_{1 \leq i, j \leq n} \longrightarrow c \det \left(\frac{c}{F_{ij}} \right)_{2 \leq i, j \leq n}. \quad (4.21)$$

The remaining prefactors involving F_{i1} and F_{1j} become the weights of the frozen column and row, so the resulting expression is exactly the freezing recurrence. The normalization and degree bound can be checked directly. We omit the remaining details. $\square$

5. ALTERNATING SIGN MATRICES

We now specialize to $\Delta = \frac{1}{2}$, where all six zero-field local weights can be made equal.

Definition 5.1. An *alternating sign matrix* of size n is an $n \times n$ matrix $A = (A_{ij})$ with entries in $\{0, 1, -1\}$ such that every row and every column has sum 1, and the nonzero entries in every row and every column alternate in sign.

For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (5.1)$$

is an alternating sign matrix. We first make precise the bijection between domain-wall six-vertex configurations and alternating sign matrices. For a horizontal edge, set

$$h = \begin{cases} +1, & \text{if the arrow points to the right,} \\ -1, & \text{if the arrow points to the left,} \end{cases} \quad (5.2)$$

and let $h_{i,j}$ denote the horizontal edge in row i immediately to the right of the j -th vertex, with $h_{i,0}$ denoting the left boundary edge. Similarly, set

$$v = \begin{cases} +1, & \text{if the arrow points upward,} \\ -1, & \text{if the arrow points downward,} \end{cases} \quad (5.3)$$

and let $v_{i,j}$ denote the vertical edge immediately above the vertex in row i , column j , with $v_{0,j}$ denoting the bottom boundary edge. Thus, the arrow configurations are encoded algebraically by the values ± 1 . The domain-wall boundary conditions give

$$h_{i,0} = 1, \quad h_{i,n} = -1, \quad v_{0,j} = -1, \quad v_{n,j} = 1. \quad (5.4)$$

These boundary values will ensure that the row and column sums of the corresponding ASM are equal to 1. At the vertex (i, j) , the ice rule is equivalent to

$$h_{i,j-1} - h_{i,j} = v_{i,j} - v_{i-1,j}. \quad (5.5)$$

Indeed, with our sign convention, the arrows pointing into the vertex contribute

$$\frac{1 + h_{i,j-1}}{2} + \frac{1 + v_{i-1,j}}{2} + \frac{1 - h_{i,j}}{2} + \frac{1 - v_{i,j}}{2}. \quad (5.6)$$

The ice rule requires exactly two incoming arrows, and simplifying this condition gives [Equation \(5.5\)](#).

Proposition 5.2. *The assignment*

$$A_{ij} := \frac{h_{i,j-1} - h_{i,j}}{2} = \frac{v_{i,j} - v_{i-1,j}}{2} \quad (5.7)$$

defines a bijection between six-vertex configurations on an $n \times n$ grid with domain-wall boundary conditions and $n \times n$ alternating sign matrices.

Proof. The equality of the two expressions in Equation (5.7) follows from Equation (5.5). Since every $h_{i,j}$ and $v_{i,j}$ is 1 or -1 , every A_{ij} belongs to $\{-1, 0, 1\}$. In particular, the entries of the ASM are determined by changes in the horizontal arrow states:

$$A_{ij} = \begin{cases} 1, & \text{if the horizontal state changes from } +1 \text{ to } -1, \\ -1, & \text{if the horizontal state changes from } -1 \text{ to } +1, \\ 0, & \text{if the horizontal state remains unchanged.} \end{cases} \quad (5.8)$$

Thus, each nonzero ASM entry records a sign change in the horizontal arrow state. Summing along row i gives

$$\sum_{j=1}^n A_{ij} = \frac{1}{2} \sum_{j=1}^n (h_{i,j-1} - h_{i,j}) = \frac{1}{2} (h_{i,0} - h_{i,n}) = 1. \quad (5.9)$$

$$\sum_{i=1}^n A_{ij} = \frac{1}{2} \sum_{i=1}^n (v_{i,j} - v_{i-1,j}) = \frac{1}{2} (v_{n,j} - v_{0,j}) = 1. \quad (5.10)$$

Along a fixed row, the sequence $h_{i,0}, h_{i,1}, \dots, h_{i,n}$ begins at $+1$ and ends at -1 . The entry A_{ij} is nonzero exactly when this sign changes. A change from $+1$ to -1 contributes $+1$, while a change from -1 to $+1$ contributes -1 . Therefore, the nonzero entries in every row alternate in sign. The same argument with the vertical edge states proves the corresponding statement for every column. Hence, A is an alternating sign matrix. Conversely, let A be an alternating sign matrix. Define³

$$h_{i,j} := 1 - 2 \sum_{k=1}^j A_{ik}, \quad v_{i,j} := -1 + 2 \sum_{k=1}^i A_{kj}. \quad (5.11)$$

Since the nonzero entries in each row and column alternate and the total sum is 1, the first and last nonzero entries must both be 1. Hence, as we read from left to right in a row, or from bottom to top in a column, the corresponding partial sums alternate between 0 and 1. Consequently, every partial row sum and every partial column sum is either 0 or 1. Hence, $h_{i,j}, v_{i,j} \in \{1, -1\}$. The row and column sum conditions give the required domain-wall boundary states. Moreover,

$$\frac{h_{i,j-1} - h_{i,j}}{2} = A_{ij} = \frac{v_{i,j} - v_{i-1,j}}{2}, \quad (5.12)$$

so the ice rule holds at every vertex. Thus, the reconstructed arrows form a unique domain-wall six-vertex configuration, and the two constructions are inverse. $\square$

Let A_n denote the number of $n \times n$ alternating sign matrices. By Proposition 5.2, A_n is obtained from the domain-wall model when all local weights are equal. Equal weights imply $\Delta = \frac{1}{2}$. From Equation (1.2), we may choose $q = e^{i\pi/3}$. Now take homogeneous line parameters

$$x_i = q^{-1}, \quad y_j = 1. \quad (5.13)$$

Then $y_j/x_i = q$, and since $q^3 = -1$,

$$a_{ij} = b_{ij} = c = q - q^{-1}. \quad (5.14)$$

³These formulas follow by inverting the relations in Equation (5.7) and applying Equation (5.4).

Every configuration therefore has weight c^{n^2} , so

$$Z_n(q^{-1}, \dots, q^{-1}; 1, \dots, 1) = c^{n^2} A_n. \quad (5.15)$$

We will use three standard facts about Schur polynomials. For a partition $\lambda = (\lambda_1, \dots, \lambda_N)$ with at most N parts,

$$s_\lambda(z_1, \dots, z_N) = \frac{\det \left(z_i^{\lambda_j + N - j} \right)_{1 \leq i, j \leq N}}{\prod_{1 \leq i < j \leq N} (z_i - z_j)}. \quad (5.16)$$

Moreover, s_λ is symmetric, its degree in each individual variable is at most λ_1 , and

$$s_\lambda(1^N) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \quad (5.17)$$

Let's motivate the expression that will give the domain-wall partition function. The starting point is Korepin's characterization in [Proposition 4.2](#). The partition function Z_n is symmetric separately in $x_1, \dots, x_n$ and $y_1, \dots, y_n$. Since $x_i^{n-1} Z_n$ is a polynomial in x_i^2 of degree at most $n-1$, and similarly for the y_j 's, it is natural to look for an expression built from the squared variables

$$x_1^2, \dots, x_n^2, y_1^2, \dots, y_n^2 \quad (5.18)$$

with the required symmetry and degree properties. Schur polynomials are natural candidates because they are symmetric in all of their variables and have a simple degree bound in each variable. We now use the freezing recurrence to determine more precisely what the Schur expression should look like. When $y_1 = x_1$, Korepin's recurrence factors Z_n into Z_{n-1} multiplied by $2n-2$ factors coming from the frozen row and column. Thus, we would like the corresponding Schur polynomial to satisfy a similar factorization under the freezing specialization. The degree bound suggests using the squared variables. We take

$$z_i = q^2 x_i^2, \quad z_{n+j} = y_j^2. \quad (5.19)$$

The factor q^2 in the first set of variables is chosen so that the freezing condition $y_1 = x_1$ becomes

$$z_{n+1} = x_1^2 = q^{-2} z_1. \quad (5.20)$$

We would like this specialization to produce linear factors of the form

$$z_k - q^2 z_1. \quad (5.21)$$

Indeed, these are, up to simple monomial factors, exactly the factors appearing in the frozen row and column. For $i \geq 2$, we have

$$q^2 x_i^2 - q^4 x_1^2 = x_1 x_i \left(q \frac{x_1}{x_i} - q^{-1} \frac{x_i}{x_1} \right), \quad (5.22)$$

while for $j \geq 2$,

$$y_j^2 - q^4 x_1^2 = q^{-1} x_1 y_j \left(q \frac{y_j}{x_1} - q^{-1} \frac{x_1}{y_j} \right). \quad (5.23)$$

Thus, we are led to look for a Schur polynomial for which the specialization

$$z_{n+1} = q^{-2} z_1 \quad (5.24)$$

produces the factors $z_k - q^2 z_1$ for all remaining variables. At $q = e^{i\pi/3}$, the quantity q^2 is a cube root of unity. If we further specialize one of the remaining variables to $z_k = q^2 z_1$, then the three relevant Schur variables are

$$z_1, \quad q^{-2} z_1, \quad q^2 z_1. \quad (5.25)$$

The values of a monomial z^a at these three points depend only on the residue class of a modulo 3. Thus, if the exponents in the numerator alternant occupy only two residue classes modulo 3,

then the three corresponding rows of the alternant lie in the span of two vectors and are therefore linearly dependent. This forces the desired zeros. This motivates the doubled staircase partition

$$\lambda^{(n)} := (n-1, n-1, n-2, n-2, \dots, 1, 1), \quad (5.26)$$

padded by two trailing zeros when regarded as a partition with at most $2n$ parts. Its alternant exponents are

$$0, 1, 3, 4, \dots, 3n-3, 3n-2, \quad (5.27)$$

so they are congruent only to 0 or 1 modulo 3. Notice also that $\lambda_1^{(n)} = n-1$, which matches the degree bound in Korepin's characterization. The resulting root-of-unity factorization is recorded in the following lemma.

Lemma 5.3 (Root-of-Unity Schur Recurrence). *Let $q = e^{i\pi/3}$. If $i \neq j$, then*

$$s_{\lambda^{(n)}}(z_1, \dots, z_{2n}) \Big|_{z_j = q^{-2}z_i} = \prod_{\substack{k=1 \\ k \neq i, j}}^{2n} (z_k - q^2 z_i) s_{\lambda^{(n-1)}}(z_1, \dots, \widehat{z_i}, \dots, \widehat{z_j}, \dots, z_{2n}), \quad (5.28)$$

where a hat indicates omission of the corresponding variable.

Proof. The proof is omitted in the talk. The main point is that the alternant exponents are congruent only to 0 or 1 modulo 3. After setting $z_j = q^{-2}z_i$, a further specialization $z_k = q^2 z_i$ makes the three corresponding rows of the numerator alternant linearly dependent. Hence, each factor $z_k - q^2 z_i$ divides the specialized Schur polynomial. These factors exhaust the available degree in z_i , and the remaining quotient is identified with $s_{\lambda^{(n-1)}}$. $\square$

At $q = e^{i\pi/3}$, the domain-wall partition function is obtained from this Schur polynomial by a monomial prefactor.

Proposition 5.4. *Let $q = e^{i\pi/3}$. Then*

$$Z_n(\mathbf{x}; \mathbf{y}) = (-1)^{n(n-1)/2} (q - q^{-1})^n \prod_{i=1}^n (qx_i y_i)^{-(n-1)} s_{\lambda^{(n)}}(q^2 x_1^2, \dots, q^2 x_n^2, y_1^2, \dots, y_n^2). \quad (5.29)$$

Proof. Let $\mathcal{S}_n(\mathbf{x}; \mathbf{y})$ denote the expression on the right-hand side. We verify [Proposition 4.2](#). For $n = 1$, the partition $\lambda^{(1)}$ is empty, so the Schur function is 1 and

$$\mathcal{S}_1 = q - q^{-1} = c. \quad (5.30)$$

The Schur function is symmetric in all of its variables, so $\mathcal{S}_n$ is separately symmetric in the x -variables and in the y -variables. Moreover, $s_{\lambda^{(n)}}$ has degree at most $n-1$ in each individual variable. The prefactor contributes $x_i^{-(n-1)}$ and $y_j^{-(n-1)}$, so $x_i^{n-1} \mathcal{S}_n$ is a polynomial in x_i^2 of degree at most $n-1$, and similarly for y_j . It remains to verify the freezing recurrence. Set $y_1 = x_1$. In the Schur variables, take

$$z_1 = q^2 x_1^2, \quad z_{n+1} = y_1^2 = x_1^2 = q^{-2} z_1. \quad (5.31)$$

By [Lemma 5.3](#), we have

$$s_{\lambda^{(n)}}(q^2 x_1^2, \dots, q^2 x_n^2, x_1^2, y_2^2, \dots, y_n^2) \quad (5.32)$$

$$= \prod_{i=2}^n (q^2 x_i^2 - q^4 x_1^2) \prod_{j=2}^n (y_j^2 - q^4 x_1^2) s_{\lambda^{(n-1)}}(q^2 x_2^2, \dots, q^2 x_n^2, y_2^2, \dots, y_n^2). \quad (5.33)$$

Since $q^3 = -1$, the factors in the first product satisfy

$$q^2 x_i^2 - q^4 x_1^2 = x_1 x_i \left(q \frac{x_1}{x_i} - q^{-1} \frac{x_i}{x_1} \right), \quad (5.34)$$

while the factors in the second product satisfy

$$y_j^2 - q^4 x_1^2 = q^{-1} x_1 y_j \left(q \frac{y_j}{x_1} - q^{-1} \frac{x_1}{y_j} \right). \quad (5.35)$$

The ratio of the monomial prefactor in $\mathcal{S}_n$ to that in $\mathcal{S}_{n-1}$ is

$$(-1)^{n-1} c (q x_1^2)^{-(n-1)} \prod_{i=2}^n (q x_i y_i)^{-1}. \quad (5.36)$$

Combining this with the factors above gives the additional scalar

$$(-1)^{n-1} q^{-3(n-1)} = 1, \quad (5.37)$$

because $q^3 = -1$. Hence,

$$\mathcal{S}_n(x_1, \dots, x_n; x_1, y_2, \dots, y_n) \quad (5.38)$$

$$= c \prod_{i=2}^n \left(q \frac{x_1}{x_i} - q^{-1} \frac{x_i}{x_1} \right) \prod_{j=2}^n \left(q \frac{y_j}{x_1} - q^{-1} \frac{x_1}{y_j} \right) \mathcal{S}_{n-1}(x_2, \dots, x_n; y_2, \dots, y_n). \quad (5.39)$$

Thus, $\mathcal{S}_n$ satisfies all of Korepin's properties. By [Corollary 4.3](#), $\mathcal{S}_n = Z_n$. $\square$

Set $x_i = q^{-1}$ and $y_j = 1$. All arguments of the Schur function in [Equation \(5.29\)](#) become 1, so

$$Z_n = (-1)^{n(n-1)/2} c^n s_{\lambda^{(n)}}(1^{2n}). \quad (5.40)$$

Since $c^2 = (q - q^{-1})^2 = -3$, comparison with [Equation \(5.15\)](#) gives

$$A_n = 3^{-n(n-1)/2} s_{\lambda^{(n)}}(1^{2n}). \quad (5.41)$$

Thus, the ASM enumeration problem has been reduced to evaluating one Schur polynomial at 1^{2n} .

Proposition 5.5 (Alternating-Sign-Matrix Enumeration). *The number of $n \times n$ alternating sign matrices is*

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \quad (5.42)$$

In particular, $A_1 = 1$, $A_2 = 2$, $A_3 = 7$, $A_4 = 42$, and $A_5 = 429$.

Proof. Pad $\lambda^{(n)}$ to length $2n$, so that

$$\lambda_{2r-1}^{(n)} = \lambda_{2r}^{(n)} = n - r, \quad 1 \leq r \leq n. \quad (5.43)$$

We apply [Equation \(5.17\)](#). For $1 \leq r < s \leq n$, set $d = s - r$. The difference between the parts in the r -th and s -th pairs is d . The four choices of one index from each pair have index differences $2d, 2d+1, 2d-1, 2d$, so the corresponding factors are

$$\frac{3d}{2d} \frac{3d+1}{2d+1} \frac{3d-1}{2d-1} \frac{3d}{2d}. \quad (5.44)$$

The factor involving the two entries within a single pair is 1. Since a fixed difference d occurs for exactly $n - d$ pairs $r < s$, we obtain

$$s_{\lambda^{(n)}}(1^{2n}) = \prod_{d=1}^{n-1} \left(\frac{9d^2(3d-1)(3d+1)}{4d^2(2d-1)(2d+1)} \right)^{n-d}. \quad (5.45)$$

Set $B_n := 3^{-n(n-1)/2} s_{\lambda(n)}(1^{2n})$, so that $A_n = B_n$. The product in Equation (5.45) is easiest to evaluate by comparing consecutive values. Dividing the product for B_n by that for B_{n-1} , every exponent decreases by one, and therefore

$$\frac{B_n}{B_{n-1}} = 3^{-(n-1)} \prod_{d=1}^{n-1} \frac{9d^2(3d-1)(3d+1)}{4d^2(2d-1)(2d+1)} = \frac{3^{n-1} \prod_{d=1}^{n-1} (3d-1)(3d+1)}{4^{n-1} \prod_{d=1}^{n-1} (2d-1)(2d+1)}. \quad (5.46)$$

The numerator contains precisely the positive integers at most $3n-2$ that are not divisible by 3, so

$$\prod_{d=1}^{n-1} (3d-1)(3d+1) = \frac{(3n-2)!}{3^{n-1}(n-1)!}. \quad (5.47)$$

For the denominator,

$$\prod_{d=1}^{n-1} (2d-1)(2d+1) = \frac{(2n-2)!(2n-1)!}{2^{2n-2}((n-1)!)^2}. \quad (5.48)$$

Substituting gives

$$\frac{B_n}{B_{n-1}} = \frac{(3n-2)!(n-1)!}{(2n-1)!(2n-2)!}. \quad (5.49)$$

Now define the proposed product

$$P_n := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \quad (5.50)$$

A direct cancellation gives

$$\frac{P_n}{P_{n-1}} = \frac{(3n-2)!(n-1)!}{(2n-1)!(2n-2)!}. \quad (5.51)$$

Thus,

$$\frac{B_n}{B_{n-1}} = \frac{P_n}{P_{n-1}}. \quad (5.52)$$

Since $B_1 = P_1 = 1$, induction gives $B_n = P_n$. Since $A_n = B_n$, we obtain Equation (5.42). $\square$

6. CONCLUSION

The six-vertex model admits several interpretations of its partition functions in terms of determinants and symmetric functions. At $\Delta = 0$, the free-fermion relation allows the model to be reformulated in terms of nonintersecting paths, leading to the LGV determinant. For generic integrable weights with domain-wall boundary conditions, the Yang-Baxter equation establishes symmetry, freezing yields a recurrence, and polynomial interpolation gives uniqueness. The resulting explicit formula is the Izergin determinant. At $\Delta = \frac{1}{2}$, the same recurrence is satisfied by a Schur polynomial at a root of unity, and the homogeneous specialization reduces the enumeration of alternating sign matrices to the evaluation of $s_{\lambda(n)}(1^{2n})$.