Week 10: The 6-vertex model, the AJS/Billey restriction formula, and puzzles

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November 18, 2026

1 Equivariant cohomology

Let $T \cong (\mathbb{C}^\times)^n$. Equivariant cohomology $H_T^\bullet$ is a contravariant functor from T-spaces to abelian groups. The diagonal map $X \to X \times X$ and the Künneth formula give a product on $H_T^\bullet(X)$, so the functor takes values in rings. In fact, pulling back along $X \to \mathrm{pt}$ makes $H_T^\bullet(X)$ an $H_T^\bullet(\mathrm{pt})$-algebra.

If $X \subseteq Y$ is a nice T-invariant subspace, we get a class $[X \subseteq Y] \in H_T^\bullet(Y)$. For example, let

$$\lambda \in T^* := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n.$$

Let $\mathbb{C}_\lambda$ be the one-dimensional representation of weight λ. The map

$$\lambda \longmapsto [\{0\} \subseteq \mathbb{C}_\lambda] \in H_T^2(\mathbb{C}_\lambda) \cong H_T^2(\mathrm{pt})$$

is an isomorphism $T^* \xrightarrow{\sim} H_T^2(\mathrm{pt})$. It turns out that

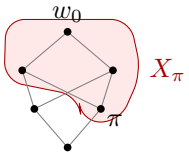
$$H_T^\bullet(\mathrm{pt}) = \mathrm{Sym}\big(H_T^2(\mathrm{pt})\big) \cong \mathbb{Z}[y_1, \ldots, y_n],$$

where each y_i has degree 2.

Now consider the flag variety $B_-\backslash G$, where B_- and B are opposite Borels with common torus T. In type A, we take $G = \mathrm{GL}_n$ and $W = S_n$. The Schubert varieties

$$X_\pi := B_-\backslash \overline{B_- \pi B} \subseteq B_-\backslash G, \qquad \pi \in W,$$

give a basis $\{[X_\pi] \mid \pi \in W\}$ for $H_T^\bullet(B_-\backslash G)$ over $H_T^\bullet(\mathrm{pt})$. The fixed points of X_π are indexed by the interval from π to w_0 in Bruhat order.



Definition 1.1. Let $\pi, \rho \in W$. The **restriction to the fixed point** ρ, denoted $[X_\pi]|_\rho$, is the pullback of $[X_\pi]$ under the inclusion

$$B_-\backslash B_-\rho \longrightarrow B_-\backslash G.$$

Equivalently, we can multiply by the class of the fixed point and integrate:

$$[X_\pi]|_\rho = \int_{B_-\backslash G} [X_\pi][B_-\backslash B_-\rho].$$

Here integration means pushforward to a point. More generally, a map $X^n \to Y^d$ between compact oriented manifolds of dimensions n and d gives a pushforward

$$H^\bullet(X) \longrightarrow H^{\bullet-(n-d)}(Y).$$

2 The AJS/Billey restriction formula

Billey proved the formula independently, and it was subsequently found in an appendix to Andersen–Jantzen–Soergel. Billey stated it for reduced words, but the formula works for an arbitrary word Q. We will use the operator version of Goldin and Knutson.

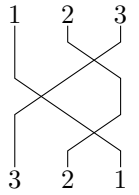
Theorem 2.1 (AJS/Billey)**.** *Pick a word $Q = r_1 \cdots r_n$ in the simple reflections of W with product $\rho = \prod Q$. Then*

$$[X_\pi]|_\rho = \sum_{\substack{P \subseteq Q \text{ a reduced subword} \\ \prod P = \pi}} \prod_{q=1}^{n} \left(\widehat{\alpha}_q^{[q \in P]} r_q\right) \cdot 1.$$

Here $1 \in H_T^0(\mathrm{pt})$. Each factor is an operator on $H_T^\bullet(\mathrm{pt})$. It applies the reflection r_q and then multiplies by its simple root α_q if the position q belongs to P. If Q is reduced, each summand is in $\mathbb{N}$[simple roots].

We can also think of the letters of the subword as divided difference operators. In cohomology the nonreduced subwords vanish in their product, so we only see reduced subwords in the formula. This point of view is useful for the analogous formulas in K-theory.

Example 2.2. Let's compute $[X_{213}]|_{321}$ in Fl(3). The permutation 213 is r_1, the reflection swapping y_1 and y_2. First use $Q = 212$.



$$\begin{aligned} [X_{213}]|_{321} &= r_2 \widehat{\alpha}_1 r_1 r_2 \cdot 1 \\ &= r_2 \cdot (y_1 - y_2) \\ &= y_1 - y_3. \end{aligned}$$

The only contributing subword is the middle letter. Here r_2 interchanges y_2 and y_3. Now use $Q = 121$. We can take the first or last letter, giving

$$\begin{aligned} [X_{213}]|_{321} &= \widehat{\alpha}_1 r_1 r_2 r_1 \cdot 1 + r_1 r_2 \widehat{\alpha}_1 r_1 \cdot 1 \\ &= (y_1 - y_2) + r_1 \cdot (y_1 - y_3) \\ &= (y_1 - y_2) + (y_2 - y_3) = y_1 - y_3. \end{aligned}$$

To see where the formula comes from, write $Q^{\leftarrow}$ for the word Q in reverse order. Using the integral description of restriction and equivariance of the pushforward, we have

$$\begin{aligned} [X_\pi]|_\rho = [X_\pi]|_{\prod Q} &= \int_{B_- \backslash G} [X_\pi]\big((\textstyle\prod Q) \cdot [B_- \backslash B_-]\big) \\ &= (\textstyle\prod Q) \cdot (\prod Q)^{-1} \cdot \int_{B_- \backslash G} [X_\pi]\big((\textstyle\prod Q) \cdot [\mathrm{pt}]\big) \\ &= (\textstyle\prod Q) \cdot \int_{B_- \backslash G} \big((\prod Q^{\leftarrow}) \cdot [X_\pi]\big)[B_- \backslash B_-]. \end{aligned}$$

Here $[\mathrm{pt}] = [B_- \backslash B_-]$. The last integral takes the coefficient of $[X_{\mathrm{id}}]$ in $(\prod Q^{\leftarrow}) \cdot [X_\pi]$, since only $[X_{\mathrm{id}}]$ restricts nontrivially at the identity.

So we need to know how a simple reflection acts on a Schubert class. For the simple root α in the restriction formula, we have

$$r_\alpha \cdot [X_\pi] = \begin{cases} [X_\pi] - \alpha [X_{\pi r_\alpha}] & \text{if } \pi r_\alpha < \pi, \\ [X_\pi] & \text{otherwise.} \end{cases}$$

Each time we take the second term, the length drops by one. Thus the choices that reach id select a reduced subword for π. On its coefficient, the outer product $\prod Q$ cancels the reflections from the reverse word. For a selected position q, the remaining action on $-\alpha_q$ is by $r_1 \cdots r_q$. Since $r_q \cdot (-\alpha_q) = \alpha_q$, this gives the factor $r_1 \cdots r_{q-1} \cdot \alpha_q$ in the restriction formula.

3 Equivariant Schubert calculus

How do we multiply two Schubert classes? We want to find the coefficients $c_{\pi\rho}^{\delta} \in H_T^\bullet(\mathrm{pt})$ in

$$[X_\pi][X_\rho] = \sum_\delta c_{\pi\rho}^{\delta}[X_\delta].$$

This is **equivariant Schubert calculus**. For every word Q we have

$$[X_\pi]|_{\prod Q}[X_\rho]|_{\prod Q} = \sum_\delta c_{\pi\rho}^{\delta}[X_\delta]|_{\prod Q}.$$

These equations determine the coefficients because the restriction map

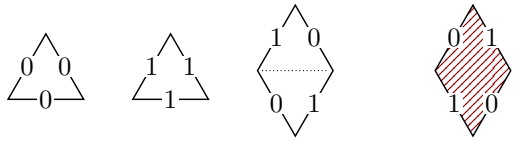
$$H_T^\bullet(B_-\backslash G) \longrightarrow H_T^\bullet\big((B_-\backslash G)^T\big) = \bigoplus_{w\in W} H_T^\bullet(\mathrm{pt})$$

is injective. Indeed, with rows indexed by π and columns indexed by ρ, the matrix of restrictions is upper triangular in a linear extension of Bruhat order, with nonzero diagonal entries.

$$\begin{array}{c} \\ \pi \end{array} \begin{array}{c} \rho \\ \left[\begin{array}{cc} [X_\pi]|_\rho & \\ [X_\pi]|_\pi \neq 0 & \\ 0 & \end{array}\right] \end{array}$$

4 Grassmannian puzzles

Definition 4.1 (Knutson–Tao, 2003)**.** A **Grassmannian puzzle** is a tiling of an equilateral triangle by the following pieces, with labels agreeing along shared edges:



up to rotation fixed orientation

The shaded rhombus is the **equivariant piece**. It has a fixed orientation.

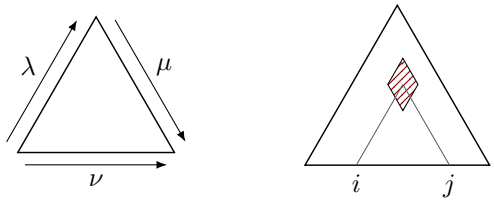
We index the Schubert classes on $\mathrm{Gr}(k,n)$ by strings of n zeros and ones with k zeros, of the form $011010\ldots1$. The positions of the zeros, in increasing order, give the first k entries of the Grassmannian permutation. The positions of the ones give the remaining entries. For example,

$$1010110 \;\longleftrightarrow\; 247\,|\,1356.$$

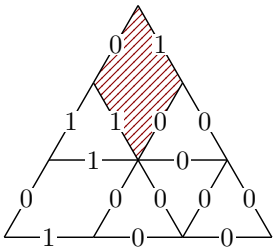
Theorem 4.2 (Knutson–Tao)**.** *The structure constants for* $\mathrm{Gr}(k,n)$ *are*

$$c_{\lambda\mu}^{\nu} = \sum_{\substack{\text{puzzles with} \\ \text{boundary } \lambda,\mu,\nu}} \prod_{\text{equivariant pieces}} (y_i - y_j).$$

Read λ *up the left side,* μ *down the right side, and* ν *from left to right along the bottom. For each shaded piece, draw the two downward diagonals from its center. If they meet the* i*th and* j*th bottom edges, from left to right, the piece contributes* $y_i - y_j$*.*

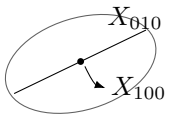


Example 4.3. Let's do $\mathbb{CP}^2 \cong \mathrm{Gr}(1,3)$. We use Grassmannian duality with $\mathrm{Gr}(2,3)$ to keep the indexing by zeros. In $[X_{010}][X_{100}]$, there is one contributing puzzle:

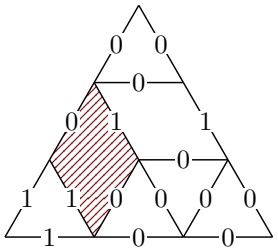


$(y_1 - y_3)[X_{100}]$.

We can also get the coefficient from the tangent cone to X_{010} at X_{100}. Here X_{010} is a line, and the coefficient is the weight in the normal direction to its tangent line at that point.

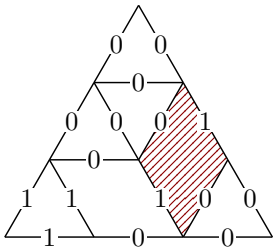


For $[X_{100}][X_{010}]$, there are two contributing puzzles:



$y_1 - y_2$

+

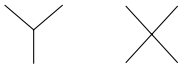


$y_2 - y_3$

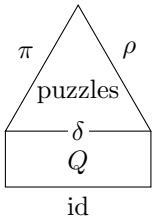
Their contributions add to $\big((y_1 - y_2) + (y_2 - y_3)\big)[X_{100}] = (y_1 - y_3)[X_{100}]$.

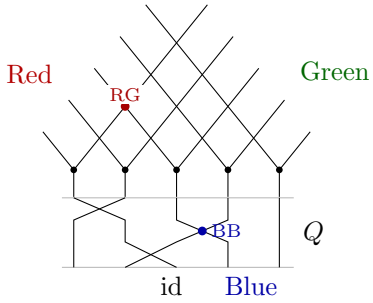
5 The 6-vertex model

We can prove the puzzle formula using the 6-vertex model. We introduce a trivalent Y-vertex alongside the X-shaped vertex of the R-matrix.

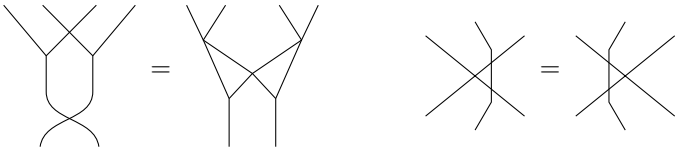


Put the puzzle triangle on top of the word Q, which we draw as a sequence of R-matrices. The lower boundary is id. Summing over the intermediate boundary δ gives the right-hand side of the restricted product formula.

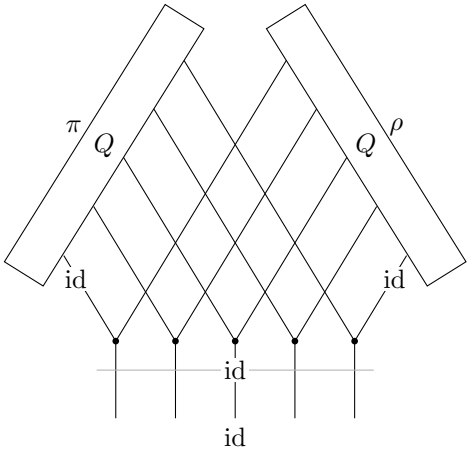




We can straighten this network using a local relation that moves a crossing past two Y-vertices. For three crossing strands we use the Yang–Baxter equation.



After moving the crossings through the network, we get the following diagram:



The two copies of Q compute the two restrictions, so the partition function is $[X_\pi]|_{\prod Q}[X_\rho]|_{\prod Q}$. Thus the puzzle coefficients satisfy the restricted product formula for every Q. By injectivity, they are the coefficients of $[X_\pi][X_\rho]$.

AI usage

AI was used for editing notes taken during the talk and generating figures and equations based on those drawn on the board.