June 29

Problem 1.

Let $V = V_1 \oplus V_2$ be a direct sum decomposition. Prove there is a unique linear map $\Pi : V \to V$ such that

(a) $\Pi(v) = v$ for $v \in V_1$, and

(b) $\Pi(v) = 0$ for $v \in V_2$.

Show that this map satisfies $\Pi^2 = \Pi$.

We can do this same procedure for the decomposition $V = V_2 \oplus V_1$. Let $\Pi^\perp$ be the map we get by doing so (pronounced "pi perp"). Prove that

$$\Pi + \Pi^\perp = 1.$$

Problem 2.

Let $W \subseteq V$ be a subspace. Show that $W^\perp$ is a subspace of V, and that

$$V = W \oplus W^\perp$$

is a direct sum decomposition.

Problem 3.

Show that

$$\langle \Pi(\mathbf{v}) \mid \Pi(\mathbf{v}) \rangle + \langle \Pi^\perp(\mathbf{v}) \mid \Pi^\perp(\mathbf{v}) \rangle = \langle \mathbf{v} \mid \mathbf{v} \rangle$$

for any orthogonal projection operator Π.

Problem 4.

Show that any linear operator on $\mathbb{C}^n$ can be written as a sum of ket-bras.