# July 22

**Problem 1.**

We will show that there is no consistent way to define a translation-invariant measure on all subsets of $\mathbb{R}$ (at least, assuming the axiom of choice). Assume that $\mu : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is such that

(i) $\mu(\varnothing) = 0$

(ii) $\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for any pairwise disjoint sequence $E_1, E_2, \ldots$ in $\mathcal{P}(\mathbb{R})$.

(iii) $\mu([0,1]) = 1$.

(iv) $\mu(x + E) = \mu(E)$ for any $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

Let $\mathbb{R}/\mathbb{Q}$ be the group quotient of $\mathbb{R}$ by its subgroup $\mathbb{Q}$; recall that

$$\mathbb{R}/\mathbb{Q} \doteq \{x + \mathbb{Q} \mid x \in \mathbb{R}\}.$$

Let $f : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$ be a function such that $f(x+\mathbb{Q}) \in x+\mathbb{Q}$ for all $x \in \mathbb{R}$. Let $E$ be the image of $f$ in $\mathbb{R}$.

(a) Show that there is a countable set $\{q_i\}_{i \in I}$ of real numbers such that

$$\bigcup_{i \in I} (q_i + E) = \mathbb{R}.$$

Conclude that $\mu(E)$ is nonzero.

(b) Let $r > 0$. Show that every coset in $\mathbb{R}/\mathbb{Q}$ intersects the interval $[0, r]$ nontrivially.

(c) Assume that $f(x+\mathbb{Q}) \in (x+\mathbb{Q}) \cap [0, \frac{1}{2}]$ for all $x$. (This is possible by part (b).) Then $E \subseteq [0, \frac{1}{2}]$. Show that there is a countably infinite set $\{q_i\}_{i \in I'}$ such that

$$\bigcup_{i \in I} (q_i + E) \subseteq [0, 1].$$

Conclude that $\mu([0,1]) = \infty$ or that $\mu(E) = 0$.

This contradiction is what motivates us to define the measure only for *measurable sets* (or, often, just for *Borel sets*). A similar construction is the origin of the Banach–Tarski paradox.

**Problem 2.**

Assume that $\mu : \mathcal{B}_{\mathbb{R}} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is such that

(i) $\mu(\varnothing) = 0$

(ii) $\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ for any pairwise disjoint sequence $E_1, E_2, \dots$ in $\mathcal{P}(\mathbb{R})$.

(iii) $\mu([0,1]) = 1$.

(iv) $\mu(x + E) = \mu(E)$ for any $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$.

Prove the following:

(a) If $E_1 \subseteq E_2 \subseteq \dots$ is an increasing sequence of Borel sets, then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sup_{n \in \mathbb{N}} \mu(E_n).$$

(b) Prove that $\mu([a,b)) = b - a$ for any $b \geq a$.

(c) Prove that $\mu((a,b)) = \mu([a,b]) = b - a$ for any $b \geq a$.

(d) Prove that any countable set is Borel and has measure 0.