July 11

Problem 1.

Let $V = \mathbb{C}^N$ have basis $|0\rangle, \dots, |N-1\rangle$, indexed by elements of $\mathbb{Z}/N\mathbb{Z}$. Define an operator T on V by

$$|x\rangle \mapsto |x-1\rangle .$$

Since the subtraction is performed in $\mathbb{Z}/N$, we have $|N\rangle = |0\rangle$. and $|-1\rangle = |N-1\rangle$.

(a) Show that T is unitary.

(b) Let $|\mathbf{v}\rangle$ be an eigenvector for T. Show that its eigenvalue λ must satisfy

$$\lambda^N = 1.$$

(c) Show that complex numbers satisfying $\lambda^N = 1$ can be identified with group homomorphisms from $\mathbb{Z}/N$ to $U(1)$ (the unitary group on $\mathbb{C}$, equivalently the group of unit complex numbers under multiplication).

(d) Show that $\langle x|\mathbf{v}\rangle = \langle 0|T^x|\mathbf{v}\rangle$ for all $x \in \mathbb{Z}/N$.

(e) Show that, for any λ satisfying $\lambda^N = 1$, that there is (up to scalar) a unique eigenvector $|\lambda\rangle$ with eigenvalue λ. If $\langle 0 \mid \lambda\rangle = 1$, then what is $\langle x \mid \lambda\rangle$ for $x \in \{0, \dots, N-1\}$?

(f) Pick a normalized eigenvector $|\lambda\rangle$ for each Nth root of unity λ. Use the spectral theorem to deduce that $\{|\lambda\rangle\}_{\lambda^N=1}$ is an orthonormal basis of V.