HW 2

Recall that if V and W are normed spaces, then the vector space $B(V,W)$ of bounded linear maps $V \to W$ comes with the *operator norm*. This is defined by

$$\|L\| \doteq \inf\{\, C \geq 0 \mid \forall \mathbf{v} \in V,\ \|L\mathbf{v}\| \leq C\mathbf{v} \,\}.$$

Note that $\|L\mathbf{v}\| \leq \|L\|\|\mathbf{v}\|$ for all $\mathbf{v} \in V$. An equivalent definition of the operator norm that is often useful is

$$\|L\| \doteq \sup\{\, \|L\mathbf{v}\| \mid \|\mathbf{v}\| \leq 1 \,\}.$$

In class we usually focus on the case where $V = W$, and we write $B(V) \doteq B(V,V)$.

Problem 1.

Prove that $B(V,W)$ is a Banach space whenever W is a Banach space. In other words, show that every Cauchy sequence of bounded linear maps $V \to W$ converges to a bounded linear map.

Definition 1.

Fix a set X and a normed space W. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions $X \to W$. We say that the sequence (f_n) *converges uniformly* to a function f if, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\|f_n(x) - f(x)\| < \varepsilon$$

for all $x \in X$. If $S \subseteq X$, then we say (f_n) *converges uniformly on* S if $(f_n|_S)$ converges uniformly.

Uniform convergence is a much stronger condition than pointwise convergence. Limit-based constructions like series, integrals, continuity, and derivatives are all preserved under uniform limits, while they may behave badly under pointwise limits. This makes uniform convergence very important for applications to calculus.

We recall the definition of Fréchet derivatives from the analysis warm-up:

Definition 2.

Let V and W be normed spaces and let X be an open subset of V. We say a function $f : X \to W$ is *(Fréchet) differentiable* at $\mathbf{v}_0 \in U$ if there exists a bounded linear map $L : V \to W$ such that

$$\lim_{\mathbf{v}\to\mathbf{v}_0} \frac{\|f(\mathbf{v}) - f(\mathbf{v}_0) - L\mathbf{v}\|}{\|\mathbf{v} - \mathbf{v}_0\|} = 0.$$

The *derivative* of f at $\mathbf{v}_0$ is the linear map L (which is unique). In this case, we say $f'(\mathbf{v}_0) = L$.

If f is differentiable at all points in X, then we say f is a *differentiable function*. In this case, let $f' : X \to B(V,W)$ be the derivative mapping. If f' is continuous (with the norm topology on $B(V,W)$), then we say that f is in $C^1(X,W)$. We inductively define $C^{n+1}(X,W)$ to consist of those functions $f \in C^n(X, W)$ such that f' is in $C^n(X, B(V, W))$. Finally, $C^\infty(X, W)$ consists of those functions in $C^n(X, W)$ for all n.

Theorem 3: Mean value theorem for Fréchet derivatives.

Let V be a normed space, W a Banach space, and X an open subset of V. If $\mathbf{v}_1, \mathbf{v}_2$ are points in V, then let $[\mathbf{v}_1, \mathbf{v}_2]$ denote the line segment between $\mathbf{v}_1$ and $\mathbf{v}_2$. Let $f \in C^1(X, W)$. Then for any points $\mathbf{v}_1, \mathbf{v}_2 \in X$ such that $[\mathbf{v}_1, \mathbf{v}_2] \subseteq X$, we have the following inequality:

$$\|f(\mathbf{v}_2) - f(\mathbf{v}_1)\| \leq \sup_{\mathbf{v} \in [\mathbf{v}_1, \mathbf{v}_2]} \|f'(\mathbf{v})\| \|\mathbf{v}_2 - \mathbf{v}_1\|.$$

Problem 2.

Parts (c) and (d) are optional. Fix a Banach space W and an open subset X of a normed space V. Let (f_n) be a sequence of functions $X \to W$ converging pointwise to a function $f : X \to W$.

(a) Fix a point $\mathbf{v}_0 \in X$ and assume that, for all n, $\lim_{\mathbf{v} \to \mathbf{v}_0} f_n(\mathbf{v})$ converges to a vector $\mathbf{w}_n \in W$. Prove that if (f_n) converges uniformly to f, then

$$\lim_{n \to \infty} \mathbf{w}_n = \lim_{\mathbf{v} \to \mathbf{v}_0} f(\mathbf{v}).$$

(b) Prove that if f_n is a continuous function for all n, and (f_n) converges uniformly to f, then f is also continuous.

(c) Show that if each f_n is in $C^1(X, W)$, and (f_n') is a uniformly convergent sequence of functions $X \to B(V, W)$, then $f \in C^1(X, W)$ and

$$\lim_{n \to \infty} f_n' = f'.$$

Conclude similarly that limits preserve a sequence of functions in $C^r(X, W)$ if $(f_n^{(r)})$ converges uniformly and all derivatives of order lower than r converge pointwise.

(d) Verify that the proofs of everything so far also work with the weaker assumption that there is a collection $\{U_i\}_{i \in I}$ of open subsets of V such that $\bigcup_{i \in I} U_i = X$ and such that, for each $i \in I$, the uniform convergence conditions hold on U_i. (As opposed to converging uniformly on all of X.)
In particular, if $X = V$, then it is sufficient to check uniform convergence on all open balls centered at the origin to conclude the results above.

If $\sum_{n=0}^\infty L_n$ is a convergent series in $B(V, W)$, then

$$L \sum_{n=0}^\infty L_n = \sum_{n=0}^\infty L L_n,$$

for any bounded linear map L, and

$$\left(\sum_{n=0}^{\infty} L_n\right)\mathbf{v} = \sum_{n=0}^{\infty} (L_n \mathbf{v})$$

for any vector $\mathbf{v} \in V$. Both of these facts follow since multiplication by L and evaluation at $\mathbf{v}$ are continuous maps on $B(V, W)$, and continuous maps preserve limits.

We now apply these results to study the operator exponential. Recall that if $L \in B(V)$ for a Banach space V, then

$$\exp(L) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} L^n.$$

Problem 3.

Let V and W be Banach spaces, and X an open subset of V.

(a) Prove the *Weirstrass M-test*. Let (f_n) be a sequence of functions $X \to W$ and (M_n) be a sequence of non-negative constants such that $\|f_n(x)\| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$. If

$$\sum_{n=0}^{\infty} M_n < \infty,$$

then the series

$$\sum_{n=0}^{\infty} f_n$$

converges uniformly (i.e., the sequence of partial sums converges uniformly).

(b) Fix a bounded linear operator L on V. Prove, using the Weirstrass M-test and Problem 2, that the function $\phi : \mathbb{R} \to B(V)$ defined by

$$\phi(t) = \exp(tL)$$

is a smooth function with derivative $\phi'(t) = L \exp(tL)$. Feel free to conflate the usual derivative with the Fréchet derivative. (Recall that they are equivalent for functions on $\mathbb{R}$.) You can also assume the usual properties of derivatives still work in the vector-valued case.

(c) Let L_1, L_2 be commuting bounded operators. Show that

$$\phi : t \mapsto \exp(t(L_1 + L_2))$$

and

$$\phi' : t \mapsto \exp(tL_1)\exp(tL_2)$$

define the same function of t. You may use that a function $\phi : \mathbb{R} \to B(V)$ with derivative 0 must be a constant function.

(d) Optional. Prove that if X is an open ball around $\mathbf{0}$ in V and $f : X \to W$ is differentiable with $f'(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in X$, then $f(\mathbf{v}) = f(\mathbf{0})$ for all $\mathbf{v} \in X$.

The rest of this assignment is dedicated to the Hilbert projection theorem and its consequences. Recall that a subset S of a vector space V is *convex* if, for any $\mathbf{v}, \mathbf{w} \in S$, every element of the line segment between $\mathbf{v}$ and $\mathbf{w}$ is also in S. That is, if $t_1, t_2 \geq 0$ satisfy $t_1 + t_2 = 1$, then $t_1\mathbf{v} + t_2\mathbf{w} \in S$. Importantly, any subspace of a vector space is convex.

Theorem 4: Hilbert projection theorem.

Let $\mathcal{H}$ be a Hilbert space. For every vector $\mathbf{v} \in \mathcal{H}$ and every closed set $C \subseteq \mathcal{H}$ which is convex, there is a unique vector $\mathbf{v}_C \in C$ for which

$$\|\mathbf{v} - \mathbf{v}_C\| = \inf_{\mathbf{w} \in C} \|\mathbf{v} - \mathbf{w}\|.$$

The vector $\mathbf{v}_C$ is called the *projection of* $\mathbf{v}$ *onto* C. To prove the projection theorem, we will use the following characterization of Hilbert spaces.

Theorem 5: Characterization of inner product spaces.

If $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, then the function $\mathbf{v} \mapsto \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$ defines a norm on V. A norm on V arises from an inner product in this way if and only if the norm satisfies the *parallelogram law*:

$$\|\mathbf{v} - \mathbf{w}\|^2 + \|\mathbf{v} + \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2$$

for all $\mathbf{v}, \mathbf{w} \in V$.
If the parallelogram law holds, then the original inner product can be determined from the *polarization identity*:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \frac{1}{4}(\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 + i\|\mathbf{v} - i\mathbf{w}\|^2 - i\|\mathbf{v} + i\mathbf{w}\|^2).$$

We won't need the polarization identity in what follows, though the parallelogram law will be quite important.

Problem 4.

Here will will prove the Hilbert projection theorem. Let $\mathcal{H}$ be a Hilbert space.

(a) Verify that proving Theorem 4 can be reduced to the case where $\mathbf{v} = 0$.

Let $C \subseteq \mathcal{H}$ be a closed convex set and define

$$d \doteq \inf_{\mathbf{w} \in C} \|\mathbf{w}\|.$$

Assume $(\mathbf{w}_n)$ is a sequence of vectors in C such that

$$\lim_{n\to\infty} \|\mathbf{w}_n\| = d.$$

(b) Show that
$$\|\mathbf{w}_n - \mathbf{w}_m\|^2 \leq 2\|\mathbf{w}_n\|^2 + 2\|\mathbf{w}_m\| - 4d^2,$$
for any $n, m \in \mathbb{N}$.

(c) Prove that $(\mathbf{w}_n)$ is a Cauchy sequence which converges to an element $\mathbf{w}$ of C such that $\|\mathbf{w}\| = d$.

Now we can already prove a uniqueness and existence statement. We know by the properties of infimums that there exists a sequence $(\mathbf{w}_n)$ of points in C such that $\lim_n \|\mathbf{w}_n\| = d$.

(d) Let $\mathbf{v}$ and $\mathbf{w}$ be two elements of C such that $\|\mathbf{v}\| = \|\mathbf{w}\| = d$. Prove that $\mathbf{v} = \mathbf{w}$. *Hint*: Consider the sequence $\mathbf{v}, \mathbf{w}, \mathbf{v}, \mathbf{w}, \ldots$.

(e) Conclude that the Hilbert projection theorem holds.

Now we can apply the projection theorem to prove fundamental results about Hilbert spaces. If C is a closed subspace of $\mathcal{H}$, then the next problem shows that the assignment $\mathbf{v} \mapsto \mathbf{v}_C$ is in fact the orthogonal projection operator onto C. (This is powerful – not every subspace comes with an orthogonal projection in the infinite-dimensional case!)

Problem 5.

Here we will show that if W is a closed subspace of a Hilbert space $\mathcal{H}$, then

$$\mathcal{H} = W \oplus W^\perp$$

is an orthogonal decomposition. Clearly the two subspaces are orthogonal. We must show that $W + W^\perp = \mathcal{H}$. Let $\mathbf{v} \in \mathcal{H}$ and let $\mathbf{v}_W$ denote the Hilbert projection of $\mathbf{v}$ onto the closed convex set W. If we can show that $\mathbf{v} - \mathbf{v}_W \in W^\perp$, then we are done.

(a) Set $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_W$. Then
$$\|\mathbf{v}_\perp\| \leq \|\mathbf{v}_\perp + \mathbf{w}\|$$
for every $\mathbf{w} \in W$, by definition of $\mathbf{v}_W$. Use this to prove that
$$\langle \mathbf{v}_\perp, \mathbf{w} \rangle = 0$$
for all $\mathbf{w} \in W$.

(b) Conclude that the orthogonal projection Π_W (which is the linear projection relative to $W \oplus W^\perp$) coincides with the map $\mathbf{v} \mapsto \mathbf{v}_W$.

Thus we can take orthogonal projections onto any closed subspace of $\mathcal{H}$. In fact, these are the only subspaces admitting orthogonal projections, since such a subspace W must satisfy $W^{\perp\perp} = W$. We will now show that $W^{\perp\perp} = \overline{W}$ for any subspace W.

(c) Show that if S is any subset of $\mathcal{H}$, then $S^{\perp}$ is a closed subspace of $\mathcal{H}$.

(d) Conclude that $\overline{W} \subseteq W^{\perp\perp}$.

(e) Show that $(\overline{S})^{\perp} = S^{\perp}$ for any subset S of $\mathcal{H}$.

(f) Conclude that $\overline{W}$ and $W^{\perp\perp}$ are both complements of $W^{\perp}$ (that is, they fit in direct sum decompositions with $W^{\perp}$). Use this with part (d) to show that $\overline{W} = W^{\perp\perp}$.

These results let us easily show (among other things) that dense orthonormal bases in a Hilbert space work just like orthonormal bases in a finite-dimensional inner product space.

Problem 6.

Let $\mathcal{H}$ be a Hilbert space with dense orthonormal basis $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, |\mathbf{e}_3\rangle, \ldots$. Show that any vector $|\mathbf{v}\rangle \in \mathcal{H}$ can be written uniquely as

$$|\mathbf{v}\rangle = \sum_{n=1}^{\infty} v_n |\mathbf{e}_n\rangle$$

for some sequence of scalars (v_n).
Hint: Let $W = \operatorname{span}\{|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots\}$. What is $W^{\perp}$?

Let's apply the projection operator results to prove some tasty theorems. The *(topological) dual space* of a normed space V is defined to be

$$V^* \doteq B(V, \mathbb{F}),$$

where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, whichever is the field of scalars. Bounded linear maps from V to the field of scalars are called *bounded (linear) functionals* or *continuous (linear) functionals*.

Problem 7. o

This problem is optional. We will prove the Riesz representation theorem, which gives a canonical isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$ for a Hilbert space $\mathcal{H}$. More precisely, if $\varphi : \mathcal{H} \to \mathbb{F}$ is any bounded functional, then the theorem asserts that there exists a unique vector $\mathbf{v}_\varphi \in \mathcal{H}$ such that

$$\langle \mathbf{v}_\varphi, \mathbf{w}\rangle = \varphi\mathbf{w}$$

for all $\mathbf{w} \in \mathcal{H}$.
The result is clear if $\varphi = 0$. So fix a bounded nonzero functional φ on $\mathcal{H}$.

(a) Show that $K \doteq \ker\varphi$ is a closed subspace of $\mathcal{H}$. By assumption K is not all of $\mathcal{H}$. Conclude using Problem 5 that $K^{\perp}$ is nonzero.

(b) Fix from here on a nonzero element $\mathbf{p}$ in $K^\perp$. Show that for any $\mathbf{w} \in \mathcal{H}$, the vector

$$(\varphi \mathbf{w})\mathbf{p} - (\varphi \mathbf{p})\mathbf{w}$$

is an element of K. As a result, the inner product of this vector with $\mathbf{p}$ is 0.

(c) Expand the equation

$$\langle \mathbf{p}, (\varphi \mathbf{w})\mathbf{p} - (\varphi \mathbf{p})\mathbf{w} \rangle = 0$$

and solve for $\varphi \mathbf{w}$. Determine a vector $\mathbf{v}_\varphi$ such that

$$\langle \mathbf{v}_\varphi, \mathbf{w} \rangle = \varphi \mathbf{w}$$

for all $\mathbf{w} \in \mathcal{H}$.

(d) Show that if $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{H}$ both satisfy

$$\langle \mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_2, \mathbf{w} \rangle = \varphi \mathbf{w}$$

for all $\mathbf{w} \in \mathcal{H}$, then $\mathbf{v}_1 = \mathbf{v}_2$.

This proves the Riesz representation theorem. It remains to show the converse, which is that the linear functional defined by a vector is in fact bounded. In fact, we can compute the norm.

(e) Let $\mathbf{v} \in \mathcal{H}$. Define a functional $\varphi : \mathcal{H} \to \mathbb{F}$ via

$$\mathbf{w} \mapsto \langle \mathbf{v}, \mathbf{w} \rangle .$$

Prove that the operator norm of φ is $\|\varphi\| = \|\mathbf{v}\|$.
Hint: Use the Cauchy–Schwartz inequality to bound the operator norm from above, and then find a point which maximizes φ to show equality.

Lastly we will prove something of great importance for this class, which is that a bounded operator on a Hilbert space has an adjoint.
Let L be a bounded linear map between Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Then for each $\mathbf{v} \in \mathcal{H}_2$, the functional

$$\varphi : \mathcal{H}_1 \to \mathbb{F}$$
$$\mathbf{w} \mapsto \langle \mathbf{v}, L\mathbf{w} \rangle$$

is continuous, since it is a composition of continuous/bounded maps. Thus the Riesz representation theorem implies there is a unique vector in $\mathcal{H}_1$, denoted $L^*\mathbf{v}$, such that

$$\varphi \mathbf{w} = \langle L^*\mathbf{v}, \mathbf{w} \rangle .$$

Definition 6.

The function $L^* : \mathcal{H}_2 \to \mathcal{H}_1$ which assigns to $\mathbf{v} \in \mathcal{H}_2$ the unique vector $L^*\mathbf{v} \in \mathcal{H}_1$ satisfying

$$\langle L^*\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, L\mathbf{w} \rangle \qquad \forall \mathbf{w} \in \mathcal{H}_1$$

is called the *adjoint* of L.

Problem 8.

Show that the adjoint of a bounded linear map L is itself a bounded linear map. Furthermore,

$$\|L^*\| = \|L\|.$$