QMMM Warm-up 2: Real analysis

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In this article, when we talk about vector spaces, the field of scalars is allowed to be either $\mathbb{R}$ or $\mathbb{C}$.

1 Topology

To talk about limits and continuity in a modern way, let's first recall some basic topology. For now and forever, if X is any set then $\mathcal{P}X$ is the power set of X (the collection of all subsets of X). If U is any subset of X, then U^c denotes the complement of U in X (equivalently, $U^c = X \setminus U$).

Definition 1.1.

Let X be any set. A *topology* τ on X is a subset of $\mathcal{P}X$ satisfying the following properties:

(1) $\varnothing, X \in \tau$

(2) If $\{U_i\}_{i\in I}$ is any collection of elements of τ, then $\bigcup_{i\in I} U_i$ is also an element of τ.

(3) If U and V are elements of τ, then $U \cap V$ is an element of τ.

Elements of τ are called *open subsets* of X (in the τ topology). A subset F of X is called a *closed subset* if F^c is an open subset. The pair (X, τ) is called a *topological space*.

Example 1.2.

Any set X can be given the *discrete topology*, which has $\tau = \mathcal{P}X$.

Usually we refer to a topological space (X, τ) just by the symbol X, and keep the topology in the back of our minds. When introducing a topological space, we might say "X is the set of all [objects], with the topology described by [description of topology]." In order to give short descriptions of the topology, we often don't want to describe every single open set. One shortcut we use is to list a few of the sets we want to be in τ, and then say "τ is the topology generated by [list of sets]" (meaning the smallest/coarsest τ containing [list of sets]). This means elements of τ are constructed from our list by repeatedly applying rules (2) and (3) in Definition 1.1. It can be very difficult to describe the sets that come out of this construction, because we may have to alternate between (2) and (3) infinitely many times. So we usually choose our list of sets so that we get a topology by only using rule (2). This motivates the following definition.

Definition 1.3.

Let X be a set and τ_0 a subset of $\mathcal{P}X$. We say τ_0 is a *base* for a topology on X if it satisfies the following:

- The sets in τ_0 *cover* X (their union is X).
- If V and W are elements of τ_0, then for any point $p \in V \cap W$, there is a set $U \in \tau_0$ such that $p \in U$ and $U \subseteq V \cap W$.

Elements of τ_0 are called *basic open subsets* of X. Define

$$\tau = \left\{ \bigcup_{i \in I} U_i \mid \{U_i\}_{i \in I} \text{ any collection of sets in } \tau_0 \right\}.$$

τ is called the *topology generated by* τ_0.

Problem 1.

Check that the "topology generated by a base" is a topology (in the sense of Definition 1.1).

In quantum mechanics and in analysis, many of the spaces that we will study are sets of real- or complex-valued functions. These spaces are vector spaces, and will come with different *norms*. A norm is a generalization of the absolute value of a number or the magnitude of a vector. As we will see below, an important feature of normed spaces is that they are automatically topological spaces (the norm induces a topology).

Definition 1.4.

Let V be a real (or complex) vector space. A *norm* on V is a function taking vectors $\mathbf{v} \in V$ to non-negative real numbers $\|\mathbf{v}\|$, satisfying the following properties for any scalar λ and any vectors $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$:

$$\begin{aligned} \|\lambda \mathbf{v}\| &= |\lambda| \ \|\mathbf{v}\| && \text{(Homogeneity)} \\ \|\mathbf{v}_1 + \mathbf{v}_2\| &\leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| && \text{(Triangle inequality)} \\ \text{If } \|\mathbf{v}\| = 0, &\text{ then } \mathbf{v} = 0. && \text{(Definiteness)} \end{aligned}$$

A vector space equipped with a choice of norm is called a *normed space*.

Example 1.5.

The *Euclidean norm* (or *2-norm*) on $\mathbb{R}^n$ or on $\mathbb{C}^n$ is given by

$$\|z_1\mathbf{e}_1 + \cdots + z_n\mathbf{e}_n\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis.

Problem 2. ♦

In the following, z^* denotes the complex conjugate of a complex number z.

(a) Prove the *GM-QM inequality*: for any $z, w \in \mathbb{C}$,

$$|z^* w| \leq \frac{|z|^2 + |w|^2}{2}.$$

(b) Prove the *Cauchy–Schwarz inequality*: for any $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$,

$$|v_1^* w_1| + |v_2^* w_2| + \cdots + |v_n^* w_n| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

(c) Prove that the Euclidean norm on $\mathbb{C}^n$ is a norm.

If $\mathbf{v}$ is a vector in a normed space V and r is a non-negative real number, the *open ball of radius r about* $\mathbf{v}$ is

$$B_r(\mathbf{v}) = \{\mathbf{w} \in V \mid \|\mathbf{w} - \mathbf{v}\| < r\}$$

and the *closed ball of radius r about* $\mathbf{v}$ is

$$\overline{B_r(\mathbf{v})} = \{\mathbf{w} \in V \mid \|\mathbf{w} - \mathbf{v}\| \leq r\}.$$

We now give V the *norm topology*: a subset U of V is open if and only if, for each point $\mathbf{v} \in U$, there is some $r > 0$ such that $B_r(\mathbf{v}) \subseteq U$. Note that the norm topology is exactly the topology generated by

$$\tau_0 = \{B_r(\mathbf{v}) \mid \mathbf{v} \in \mathbb{R}^n,\ r > 0\}.$$

Problem 3.

Fix $n \in \mathbb{N}$. Prove that the collection

$$\tau_0 = \{B_r(\mathbf{v}) \mid \mathbf{v} \in V,\ r > 0\}$$

is a base for a topology on V. This implies via Problem 1 that the norm topology on V is indeed a topology.

Problem 4. o

Check that the following sets have the properties we expect.

(a) If V is a normed space, then $\overline{B_r(\mathbf{v})}$ is a closed set for all $r \geq 0$ and $\mathbf{v} \in V$.

(b) The open box $(-r, r)^2$ is open in $\mathbb{R}^2$.

(c) The closed box $[-r, r]^2$ is closed in $\mathbb{R}^2$.

A *neighborhood* of a point p in a topological space X is a set $S \subseteq X$ such that there exists an open set $U \subseteq S$ with $p \in U$. An *open neighborhood* of p is a neighborhood which is open (so, an open set containing p). A *closed neighborhood* of p is a neighborhood which is closed.

Definition 1.6.

A topological space X is *Hausdorff* if, for any distinct pair of points p and q, there are open neighborhoods U, V of p and q, respectively, such that $U \cap V = \varnothing$.

A subset S of a topological space X is called a *compact set* if for any collection of open sets $\{U_i\}_{i \in I}$ such that $S \subseteq \bigcup_{i \in I} U_i$, there exists a finite set $I' \subseteq I$ such that

$$S \subseteq \bigcup_{i \in I'} U_i.$$

Intuitively speaking, a Hausdorff space is one which doesn't have any points which are glued infinitely close to one another. The Hausdorff condition is exactly what makes limits of sequences or functions unique. Compactness encapsulates the notion of a set in which there is "not much room". In nice spaces (like normed spaces), any infinite subset of a compact set S must get infinitely close to at least one point in S. We'll see more about these interpretations in the next section. First, we prove some basic properties of normed spaces and compact sets.

Problem 5.

(a) Prove that any normed space (with the norm topology) is Hausdorff.

(b) We say a subset S of a normed space is *bounded* if there exists an $r > 0$ such that for any $\mathbf{v}, \mathbf{w} \in S$, we have $\|\mathbf{v} - \mathbf{w}\| \leq r$. Prove that a set S is bounded if and only if there exists an $r' > 0$ such that $S \subseteq \overline{B_{r'}(\mathbf{0})}$.

(c) Prove that if S is a compact subset of a normed space, then S is bounded.

Problem 6.

(a) Prove that a compact subset of a Hausdorff topological space is a closed set.

(b) Prove that a closed subset of a compact set is compact (in any topological space).

Problem 5 and Problem 6 (a) show that a compact subset of a normed space must be closed and bounded. In finite dimensions, the converse also holds.

Theorem 1.7: Heine–Borel Theorem.

Let $V = \mathbb{R}^n$ have the Euclidean norm topology. Then a subset S of V is compact if and only if S is closed and bounded.

Proof. As noted above, we have already shown that compact sets are closed and bounded. So assume that S is a closed and bounded subset of V. By Problem 5 (b), S is contained in $\overline{B_r(\mathbf{0})}$ for some $r > 0$. If we can show that $\overline{B_r(\mathbf{0})}$ is compact, it will follow from Problem 6 (b) that S is compact as well. It will be easiest to make one more simplification:

Problem 7.

Show that $\overline{B_r(\mathbf{0})}$ is contained in $[-r, r]^n$.

In Problem 27, we will show that $[-r, r]^n$ is compact in the *product topology* on $\mathbb{R}^n$. In Problem 29, we will show that the product topology is the same as the Euclidean norm topology. Thus S is a closed subset of the compact set $[-r, r]^n$, so S is compact. □

The Heine–Borel theorem also holds for any norm on $\mathbb{R}^n$; this follows from Problem 29 (d) and (e). A topological space X is called *locally compact* if every point of X has a compact neighborhood.

Problem 8.

Prove, using the Heine–Borel Theorem, that a finite dimensional normed space is locally compact.

1.1 Limits and cluster points

Definition 1.8.

Let X be a topological space. If $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in X, then we say that a point x is a *limit point* of (x_n) if, for every open neighborhood U of x, there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. The point x is a *cluster point* of (x_n) if, for every open neighborhood U of x and for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $x_n \in U$.

If S is a subset of X, then we say that a point $x \in X$ is an *adherent point* of S if, for every open neighborhood U of x, there is an element of S which is in U. The *closure of* S is the set

$$\overline{S} \doteq \{\, x \in X \mid x \text{ is an adherent point of } S \,\}.$$

An *accumulation point* of S is an adherent point of $S \setminus \{x\}$. (Confusingly, this is often also called a *cluster point* of S.)

Problem 9.

Let S, T be subsets of a topological space X. Prove the following properties of the set closure.

(a) $\overline{S}$ is closed.

(b) If F is a closed set containing S, then $\overline{S} \subseteq F$.

(c) The map $S \mapsto \overline{S}$ is a closure operator. In other words, it satisfies the following properties:

   (i) $S \subseteq \overline{S}$.

   (ii) If $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

   (iii) $\overline{\overline{S}} = \overline{S}$.

(d) $\overline{\varnothing} = \varnothing$.

(e) $\overline{S \cup T} = \overline{S} \cup \overline{T}$.

Problem 10.

Prove that in a Hausdorff topological space, each sequence has at most one limit point.

Thus if our space is Hausdorff (e.g., a normed space), then sequences have a unique limit point if one exists. In this case, if the sequence (x_n) has a limit point, then we say (x_n) *is convergent.* If x is the unique limit point of (x_n), then we say *x is the limit of (x_n)* or that (x_n) *converges to x*, and we write $(x_n) \to x$ or $\lim_{n\to\infty}(x_n) = x$. Note that cluster points need not be unique, even in a Hausdorff space!

Example 1.9.

If we are in a normed space, then to check that x is a limit, cluster, or adherent point it is sufficient to check the appropriate property only for those open neighborhoods which are open balls centered at x. In other words:

> A sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ of vectors in a normed space converges to a vector $\mathbf{v}$ if and only if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
>
> $$\mathbf{v}_n \in B_\varepsilon(\mathbf{v}).$$

Similarly,

> A vector $\mathbf{v}$ in a normed space is a cluster point of the sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ if and only if, for all $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists an $n \geq N$ such that
>
> $$\mathbf{v}_n \in B_\varepsilon(\mathbf{v}).$$

For adherent points, the condition is just

> In a normed space, the vector $\mathbf{v}$ is an adherent point of the set S if and only if, for all $\varepsilon > 0$, there exists an element of S in $B_\varepsilon(\mathbf{v})$.

Problem 11.

Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in a normed space.

(a) Prove that if $(\mathbf{v}_n)$ converges, then $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is a bounded set.

(b) Prove that if $(\mathbf{v}_n)$ converges, then any subsequence of $(\mathbf{v}_n)$ converges to the same limit.

(c) Prove that $\mathbf{v}$ is a cluster point of $(\mathbf{v}_n)$ if and only if there exists a subsequence $(\mathbf{v}_{n_k})_{k\in\mathbb{N}}$ which converges to $\mathbf{v}$. (This fails for an arbitrary topological space!)

The previous problem shows that a convergent sequence in a normed space is bounded and has a unique cluster point. In fact, the converse holds in finite dimensions. To prove this, we use the *Bolzano—Weierstrass* theorem.

Theorem 1.10: Bolzano–Weierstrass Theorem.

If $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is a bounded set in $\mathbb{R}^n$ (or $\mathbb{C}^n$), then some subsequence of $(\mathbf{v}_n)_{n\in\mathbb{N}}$ is convergent.

Proof. Since $\{\mathbf{v}_n\}$ is bounded, it is a subset of $\overline{B_r(\mathbf{0})}$ for some $r > 0$ by Problem 5 (b). By the Heine–Borel Theorem, $\overline{B_r(\mathbf{0})}$ is compact. The proof of our theorem now follows from Problem 11 (c) and the following problem.

Problem 12.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in a compact subset S of a topological space X. Prove that (x_n) has a cluster point in S.

$\square$

Problem 13.

Show that a sequence in a finite dimensional normed space is convergent if and only if it is bounded and has a unique cluster point.

1.2 Continuity

Definition 1.11.

Let X and Y be topological spaces and let $f : X \to Y$ be some function. We say that f is a *continuous map* if, for every open subset U of Y, the preimage $f^{-1}U$ is open in X.

We say that f is an *open map* if, for every open subset U of X, the image $f(U)$ is open in Y. Similarly, f is a *closed map* if the image of every closed set in X is closed in Y.

A continuous open bijection is called a *homeomorphism*.

Problem 14.

Let S be a compact subset of a topological space X. Prove that if $f : X \to Y$ is a continuous map, then the image $f(S)$ is a compact subset of Y.

Problem 15. o

Let X be a topological space such that X is compact, and let Y be a Hausdorff space. Prove that any continuous map $f : X \to Y$ is a closed map. If, additionally, f is surjective, then show that f is also an open map.

Problem 16.

Let $f : X \to Y$ be a continuous function and let (x_n) be a sequence in X which has a limit point $x \in X$. Show that $f(x)$ is a limit point of $(f(x_n))$. Conclude that if X and Y are Hausdorff, then continuous functions take limits to limits.

Example 1.12.

Like with limits, continuity of a map between normed spaces can be checked on open balls. This gives the usual ε-δ formulation of continuity.

> If V and W are normed spaces, then a function $f : V \to W$ is continuous if and only if, for each $\mathbf{v} \in V$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that
>
> $$B_\delta(\mathbf{v}) \subseteq f^{-1} B_\varepsilon(f(\mathbf{v})).$$
>
> Equivalently,
>
> $$f B_\delta(\mathbf{v}) \subseteq B_\varepsilon(f(\mathbf{v})).$$
>
> Equivalently,
>
> $$\text{If } \mathbf{v}' \text{ is such that } \|\mathbf{v}' - \mathbf{v}\|_V < \delta, \text{ then } \|f(\mathbf{v}') - f(\mathbf{v})\|_W < \varepsilon.$$

Problem 17.

Let f, g be continuous maps from a topological space X to a normed space V. Prove the following. (Feel free to assume X is a normed space for concreteness.)

(a) If λ is any scalar, then λf is continuous.

(b) $f + g$ is continuous.

(c) If $h : V \to V$ is continuous, then $h \circ f$ is continuous.

(d) If $V = \mathbb{C}$, then f^2 (mapping $x \in X$ to $f(x)^2$) is continuous.

(e) If $V = \mathbb{C}$, then fg (mapping x to $f(x)g(x)$) is continuous.

(f) If $V = \mathbb{C}$ and $f(x) \neq 0$ for all $x \in X$, then $\frac{1}{f}$ is continuous.

1.3 Completeness

Definition 1.13.

Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence of points in a normed space. We say $(\mathbf{v}_n)$ is a *Cauchy sequence* if, for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that for any $n, m \geq N$,

$$\|\mathbf{v}_n - \mathbf{v}_m\| < \varepsilon.$$

Problem 18. ♦

Prove that any convergent sequence in a normed space is a Cauchy sequence.

Let V be a normed space. If every Cauchy sequence in V is a convergent sequence, then we say that V is *(Cauchy) complete*. A complete normed space is also called a *Banach space*. I think now is a good time to recall the axioms defining the real numbers. This is usually described in terms of the *least upper bound property*. We will show that this implies Cauchy completeness of $\mathbb{R}$.

Definition 1.14.

Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say a number $M \in \mathbb{R}$ is an *upper bound* for S (written $S \leq M$) if every element $s \in S$ satisfies $s \leq M$. A *supremum* (or *least upper bound*) of S is an upper bound M such that, for any upper bound M' of S, we have $M \leq M'$.
Lower bounds and infimums (greatest lower bounds) are defined similarly.

Definition 1.15.

$\mathbb{R}$ is characterized by the following properties:

(1) $\mathbb{R}$ is an ordered field, meaning that $\mathbb{R}$ is a field with a total order satisfying

$$x < y \implies x + z < y + z$$

and

$$x > 0 \text{ and } y > 0 \implies xy > 0$$

for all $x, y, z \in \mathbb{R}$.

(2) $\mathbb{R}$ has the least upper bound property: any nonempty set of real numbers with an upper bound has a least upper bound in $\mathbb{R}$.

The unique supremum of a bounded set S of real numbers is denoted $\sup S$. If S is unbounded above, we write $\sup S = \infty$. Similarly, the infimum of S is denoted $\inf S$, and may be equal to $-\infty$.

Problem 19.

Prove the following properties of the real numbers.

(a) Any nonempty set which is bounded below has an infimum.

(b) If S, T are sets of real numbers, then define

$$S + T \doteq \{\, s + t \mid s \in S,\ t \in T \,\}.$$

Prove that $\sup(S + T) = \sup(S) + \sup(T)$.

(c) $\mathbb{R}$ satisfies the *archimedean property*: if $x, y \in \mathbb{R}$ satisfies

$$x \leq y < x + \frac{1}{n}$$

for all $n \in \mathbb{N}$, then $y = x$.

Problem 20.

Show that a real number M is the supremum of a set S if and only if M is an upper bound for S and there is a sequence of elements in S which converges to M.

Theorem 1.16: Heine–Borel for $\mathbb{R}$.

The closed ball $\overline{B_r(\mathbf{0})} = [-r, r]$ is a compact subset of $\mathbb{R}$.

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of $[-r, r]$. We want to show that there is a finite subcollection of $\{U_i\}$ which covers $[-r, r]$. Define the set

$$S \doteq \{\, x \in [-r, r] \mid [-r, x] \text{ is covered by a finite subcollection of } \{U_i\} \,\}.$$

Note that S is nonempty, since $-r \in S$. Let M be the supremum of S. If we can show that

(a) M is actually contained in S, and

(b) $M = r$,

then we will have proven that $[-r, r]$ is compact. Let us first prove (a). Since r is an upper bound for S, we know that $M \leq r$. We also know $M \geq -r$, since $-r \in S$. So M is in $[-r, r]$, and therefore is contained in some open set $U \in \{U_i\}$, because $\{U_i\}$ covers $[-r, r]$. Since U is open, it contains some open interval $(M - \varepsilon, M + \varepsilon)$ with $\varepsilon > 0$.

Pick a point $x \in S \cap (M - \varepsilon, M]$; such a point must exist, or else $M - \frac{\varepsilon}{2}$ is an upper bound for S which is lower than M. By definition of S, we know that $[-r, x]$ is covered by finitely many of the sets in $\{U_i\}$. Observe that

$$[-r, M] \subseteq [-r, x] \cup U.$$

Hence, to cover $[-r, M]$, we need only one set in addition to the finitely many sets covering $[-r, x]$. This proves (a).

To show (b), assume that M is strictly less than r. We know that $[-r, M]$ is contained in the union of finitely many elements from $\{U_i\}$. Call this union V. Then V is an open set containing M, so it also contains an interval $(M - \varepsilon, M + \varepsilon)$ for some $\varepsilon > 0$, small enough so that $M + \varepsilon \leq r$. But that means that

$$[-r, M + \frac{\varepsilon}{2}]$$

is contained in V as well, which implies that $M + \frac{\varepsilon}{2} \in S$. But this contradicts the fact that M is an upper bound for S, which is absurd. So it must be the case that $M = r$, and (b) holds. We have shown that $[-r, r]$ is covered by finitely many of the sets $\{U_i\}$. Since this holds for an arbitrary open cover, it follows that $[-r, r]$ is compact. $\square$

Definition 1.17.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. The *limit supremum* of (x_n) is defined by

$$\limsup_{n\to\infty} x_n \doteq \inf\{\, \sup\{\, x_n \mid n \geq N \,\} \mid N \in \mathbb{N} \,\} = \inf_{N \in \mathbb{N}} \sup_{n \geq N} x_n.$$

The *limit infimum* of (x_n) is

$$\liminf_{n\to\infty} x_n \doteq \sup\{\, \inf\{\, x_n \mid n \geq N \,\} \mid N \in \mathbb{N} \,\} = \sup_{N \in \mathbb{N}} \inf_{n \geq N} x_n.$$

(Note that the various supremums and infimums appearing in these definitions may be $\pm\infty$.)

Problem 21.

Prove the following about a sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers.

(a) $\limsup_{n\to\infty} x_n = \lim_{N\to\infty} \sup_{n \geq N} x_n$ and $\liminf_{n\to\infty} x_n = \lim_{N\to\infty} \inf_{n \geq N} x_n$.

(b) $\limsup_{n\to\infty} x_n \geq \liminf_{n\to\infty} x_n$.

(c) (x_n) is a convergent sequence if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x \in \mathbb{R}.$$

In this case, $\lim(x_n) = x$.

Problem 22. ♦

Prove that $\mathbb{R}$ is Cauchy complete: any Cauchy sequence in $\mathbb{R}$ is a convergent sequence.

Aside: Subspaces and products

Definition 1.18.

Let S be a subset of a topological space X. Then S inherits a topology, the *subspace topology* (induced by X), described as follows: a subset $U' \subseteq S$ is open if and only if there exists an open subset U of X such that $U \cap S = U'$.

Problem 23. o

Check the following for a subset S of a topological space X, with the subspace topology.

(a) The subspace topology is a topology. If X is Hausdorff, then S is Hausdorff.

(b) A subset F' of S is closed in S if and only if there exists a closed subset F of X such that $F \cap S = F'$.

(c) If T is a subset of S, then the subspace topology on T induced by S is the same as the subspace topology on T induced by X.

(d) If $f : X \to Y$ is a continuous map, then the restriction $f|_S$ is a continuous map $S \to Y$.

Problem 24. o

A topological space X is called compact if X is a compact subset of X. Check that a subset S of X is compact if and only if S with the subspace topology is a compact space.

Recall that $X \times Y$ denotes the (Cartesian) product of X and Y, is the set of all ordered pairs from X and Y:

$$X \times Y \doteq \{ (x, y) \mid x \in X,\ y \in Y \} .$$

If $S \subseteq X$ and $T \subseteq Y$, then $S \times T$ is a subset of $X \times Y$.

Definition 1.19.

Let X and Y be two topological spaces. This set comes with an induced topology, the *product topology*. This topology has a base given by

$$\tau_0 = \{ U \times V \mid U \text{ is a basic open subset of } X,\ V \text{ is a basic open subset of } Y \} .$$

Problem 25. o

Check that τ_0 is a base for a topology on $X \times Y$, and the topology generated by τ_0 is independent of the bases used for X and Y.

Problem 26.

Let S and T be compact subsets of topological spaces X and Y, respectively. Prove that $S \times T$ is a compact subset of $X \times Y$.

Using these definitions, we can give $\mathbb{R}^n$ the product topology, which has a base consisting of products of open intervals in $\mathbb{R}$. Note that it is not obvious that this gives the same topology as the Euclidean norm topology which we have previously studied (though in fact it does). However, it is more obviously a topology for a norm, as we show below.

Problem 27.

Define a function $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ which selects the largest coordinate of a vector:

$$\|\mathbf{v}\|_\infty \doteq \max_{1 \le k \le n} |v_k|.$$

Show that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$. (This is the ∞-*norm* on $\mathbb{R}^n$.)
Prove that the product topology on $\mathbb{R}^n$ is the same as the norm topology using $\|\cdot\|_\infty$.

It is a hugely important theorem that *every norm on* $\mathbb{R}^n$ *induces the same topology.* We are now well-equipped to prove this theorem.

Problem 28.

Prove the following.

(a) The set $[-r, r]^n$ is compact in the product topology on $\mathbb{R}^n$.

(b) The set
$$\{\, \mathbf{v} \in \mathbb{R}^n \mid \|\mathbf{v}\|_\infty = 1 \,\}$$
is compact in the product topology on $\mathbb{R}^n$ (equivalently, the ∞-norm topology).

Problem 29. ♦

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be any two norms on $\mathbb{R}^n$. Prove the following.

(a) There is some $C > 0$ such that

$$\|\mathbf{v}\|_a \leq C\|\mathbf{v}\|_\infty$$

for all $\mathbf{v}$ in $\mathbb{R}^n$.

(b) Any norm is a continuous function from $\mathbb{R}^n$ (with the ∞-norm topology) to $\mathbb{R}$.

(c) There exists a constant $C > 0$ such that

$$\|\mathbf{v}\|_a \leq C\|\mathbf{v}\|_b$$

for all $\mathbf{v} \in \mathbb{R}^n$.

(d) The norm topology on $\mathbb{R}^n$ induced by $\|\cdot\|_a$ is the same as the norm topology induced by $\|\cdot\|_b$.

(e) The bounded sets in $\mathbb{R}^n$ with the $\|\cdot\|_a$ norm are the same as the bounded sets with the $\|\cdot\|_b$ norm.

Problem 30. ♦

Prove that every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

1.4 Series

Definition 1.20.

Let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in a normed space V. We say that *the series* $\sum_{n\in\mathbb{N}} \mathbf{v}_n$ *converges* if the sequence of partial sums

$$\mathbf{s}_n = \sum_{k=0}^{n} \mathbf{v}_k$$

converges to a limit $\mathbf{s}$. In this case, the limit is unique, and we say that $\mathbf{s}$ is *the sum of the series*, written

$$\sum_{n\in\mathbb{N}} \mathbf{v}_n = \sum_{n=0}^{\infty} \mathbf{v}_n = \mathbf{s}.$$

Problem 31. ♦

Let V be a Banach space and let $(\mathbf{v}_n)_{n\in\mathbb{N}}$ be a sequence in V. Prove that if

$$\sum_{n\in\mathbb{N}} \|\mathbf{v}_n\|$$

converges to a real number, then

$$\sum_{n\to\infty} \mathbf{v}_n$$

is convergent.

A series $\sum_n \mathbf{v}_n$ such that $\sum_n \|\mathbf{v}_n\|$ converges is called *absolutely convergent*. The previous problem shows that absolute convergence implies convergence in Banach spaces, generalizing the familiar property of series in $\mathbb{R}$. (In fact, in a normed space, this property is equivalent to being Banach. This observation will ultimately be how we construct examples of Banach spaces in the second week of class.)

1.5 Limits, differentiation and function spaces

Definition 1.21.

Let V and W be normed spaces, and X, Y be subsets of V and W, respectively. Let $f : X \to Y$ be any function, and $\mathbf{v}_0 \in V$ be any accumulation point of X. We say that L is the *limit as* $\mathbf{v}$ *approaches* $\mathbf{v}_0$ *of* f if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{v} \in X$,

$$\|f(\mathbf{v}) - L\| < \varepsilon \text{ whenever } 0 < \|\mathbf{v} - \mathbf{v}_0\| < \delta.$$

If $\mathbf{v}_0 \in V$, we say f is *continuous at the point* $\mathbf{v}_0$ if $f(\mathbf{v}_0)$ is the limit of f as $\mathbf{v}$ approaches $\mathbf{v}_0$.

Problem 32.

Check that limits of functions are unique when they exist. Prove that $f : X \to Y$ is continuous iff it is continuous at every accumulation point of X which is in X.

Definition 1.22.

Let V and W be normed spaces and let U be an open subset of V. We say a function $f : U \to W$ is *(Fréchet) differentiable* at $\mathbf{v}_0 \in U$ if there exists a continuous linear map $L : V \to W$ such that

$$\lim_{\mathbf{v}\to\mathbf{v}_0} \frac{\|f(\mathbf{v}) - f(\mathbf{v}_0) - L\mathbf{v}\|}{\|\mathbf{v} - \mathbf{v}_0\|} = 0.$$

The *derivative* of f at $\mathbf{v}_0$ is the linear map L (which is unique). In this case, we say $f'(\mathbf{v}_0) = L$.

If f is differentiable at all points in U, then we say f is a *differentiable function.*

We will mostly be interested in the case where $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$. In this case, via Problem 30, every linear map is continuous, so the continuity condition can be dropped. Additionally, the vector space of linear maps $\mathbb{R}^n \to \mathbb{R}^m$ can be identified with $\mathbb{R}^{nm}$, the vector space of $n \times m$ matrices. Since this is finite-dimensional, it has a unique norm topology by Problem 29. We say that a differentiable function f is *continuously differentiable* if the map $f' : V \to \mathbb{R}^{nm}$ is continuous using the norm topologies. We use the notation

$$C^1(V;W) \doteq \{\, f : V \to W \mid f \text{ is continuously differentiable} \,\}\,.$$

Similarly, $C^0(V;W)$ or $C(V;W)$ denotes the set of continuous functions, $C^n(V;W)$ denotes the set of functions whose nth derivative is continuous, and $C^\infty(V;W)$ denotes the set of smooth (infinitely differentiable) functions. If the codomain $W = \mathbb{R}$, we will usually just write $C^n(V)$ for $C^n(V;\mathbb{R})$.

Remark 1.23. If you are familiar with complex analysis, note that Definition 1.22 is *not* the same as complex differentiability, even when f is a function from $\mathbb{C}$ to $\mathbb{C}$. Instead, this definition views $\mathbb{C}$ as $\mathbb{R}^2$ and checks real differentiability (which is much weaker).

Problem 33. ⧫

Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is Fréchet differentiable if and only if it is differentiable in the usual sense. How does the the Fréchet derivative (a function from $\mathbb{R}$ to the space of linear maps $\mathbb{R} \to \mathbb{R}$) correspond to the usual derivative (a function from $\mathbb{R}$ to $\mathbb{R}$)?

Definition 1.24.

Let $V = \mathbb{R}^n$ and $W = \mathbb{R}$ and $f : V \to W$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of V. The *kth partial derivative of f at* $\mathbf{v} \in V$ is

$$\frac{\partial f}{\partial x_k}(\mathbf{v}) = \lim_{h \to 0} \frac{f(\mathbf{v} + h\mathbf{e}_k) - f(\mathbf{v})}{h}.$$

Here the limit is taken as the real number h approaches 0, as in the usual definition of a derivative.

As an extension of the last problem, one can show the following.

Theorem 1.25.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then $f \in C^1(\mathbb{R}^n)$ if and only if each of its n partial derivatives exist and are continuous functions $\mathbb{R}^n \to \mathbb{R}$.

1.6 Integrals

We will ultimately only be interested in Lebesgue integrals, which are more general than the Riemann integrals you may have studied before. Because of this, we won't commit to any specific implementation of integration at the moment. Here we present an interesting axiomatic description[1] of the integral of a continuous function. Recall that $C(\mathbb{R})$ denotes the set of continuous functions $\mathbb{R} \to \mathbb{R}$.

[1] The original source for this is a document I found online called "Axioms for Integration", which unfortunately does not have a named author.

Definition 1.26.

A *definite integral* is a function which takes as input a real number a, a real number b, and a function $f \in C(\mathbb{R})$, and outputs a real number denoted

$$\int_a^b f(x)\ dx.$$

Furthermore, a definite integral must satisfy the following properties, for all $a, b, c, \lambda \in \mathbb{R}$ and $f, g \in C(\mathbb{R})$.

$$\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx \qquad \text{(Linearity I)}$$

$$\int_a^b \lambda f(x)\ dx = \lambda \int_a^b f(x)\ dx \qquad \text{(Linearity II)}$$

$$\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx \qquad \text{(Additivity)}$$

$$\text{If } a \leq b, \text{ and } f(x) \geq 0 \text{ for all } x \in [a, b], \text{ then } \int_a^b f(x)\ dx \geq 0 \qquad \text{(Positivity)}$$

$$\int_a^b 1\ dx = b - a. \qquad \text{(Normalization)}$$

These are all surely properties that we expect a definite integral to have. You might check that your favorite integral definition satisfies these properties. In fact, we will show that these properties uniquely determine a function

$$\mathbb{R} \times \mathbb{R} \times C(\mathbb{R}) \longrightarrow \mathbb{R},$$

so these properties fully encapsulate the integration of continuous functions. The benefit of studying Riemann or Lebesgue integrals in depth is that they satisfy the properties above even for non-continuous functions, which we will need to use in order to prove completeness theorems in quantum mechanics.

Problem 34.

Prove the following.

(a) $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$

(b) If $a \leq b$, and $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx.$$

(c) The triangle inequality for integrals: if $a \leq b$ and $f \in C(\mathbb{R})$, then

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)|\ dx.$$

The fundamental theorem of calculus is now an easy corollary of the properties in Definition 1.26.

Problem 35.

Let $f \in C(\mathbb{R})$ and $a \in \mathbb{R}$. Define a function $F : \mathbb{R} \to \mathbb{R}$ via

$$F(x) \doteq \int_a^x f(t)\ dt.$$

Prove the following properties of F.

(a) There is a constant $C > 0$ such that

$$|F(x) - F(y)| \leq C|x - y|.$$

(b) F is a continuous function.

(c) If $c \in \mathbb{R}$, then

$$\lim_{x \to c} \frac{\int_c^x |f(t) - f(c)|\ dt}{x - c} = 0.$$

(d) F is a differentiable function satisfying $F'(x) = f(x)$ for all $x \in \mathbb{R}$.

This fundamental theorem of calculus says that the definite integral of a continuous function gives an antiderivative of that function. One can use the mean value theorem to show that an antiderivative of $f \in C(\mathbb{R})$ is determined by its value at a point — as a result, the definite integral of f is unique, since we know $F(a) = 0$ from

$$F(a) = \int_a^a f(x)\ dx = -\int_a^a f(x)\ dx = -F(a).$$

Appendix: More norms

Here we study a family of norms which is quite useful.

Definition 1.27.

Let $p \geq 1$ be a real number (or ∞). We define the *p-norm* on $\mathbb{R}^n$ to be the following function.

$$\|\cdot\|_p : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\|v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n\|_p \doteq \left(|v_1|^p + |v_2|^p + \cdots + |v_n|^p\right)^{1/p}.$$

Here $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denotes the standard basis of $\mathbb{R}^n$.

If $p = \infty$, then we define $\|\mathbf{v}\|_\infty = \max\limits_{1 \leq k \leq n} |v_k|$, as in the earlier sections.

Problem 36. o

In the following, let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.
You may make use of *Jensen's inequality*, which states: if $f \in C^2([a,b])$ and $f''(x) \geq 0$ for all $x \in (a,b)$, then

$$f(t_1 a + t_2 b) \leq t_1 f(a) + t_2 f(b)$$

for all $0 \leq t_1, t_2 \leq 1$ such that $t_1 + t_2 = 1$.

(a) Prove *Young's inequality*: for all $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(b) Prove *Hölder's inequality*: for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$,

$$|v_1^* w_1| + |v_2^* w_2| + \cdots + |v_n^* w_n| \leq \|\mathbf{v}\|_p \|\mathbf{w}\|_q.$$

(c) Prove *Minkowski's inequality* (the triangle inequality for the p-norm): for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$,

$$\|\mathbf{v} + \mathbf{w}\|_p \leq \|\mathbf{v}\|_p + \|\mathbf{w}\|_p.$$

(d) Show that the p-norm is a norm.