Infinite root systems in algebra and geometry

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Grant T. Barkley

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Dissertation Advisor: Lauren K. Williams Grant T. Barkley

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Abstract

Given a root system, we study sets of roots, called biclosed sets, introduced by Matthew Dyer to generalize the Kazhdan–Lusztig conjecture. Biclosed sets generalize the inversion sets of Weyl group elements. Most of their properties are standard for finite root systems and completely conjectural for infinite root systems. We prove several conjectures of Dyer about biclosed sets for affine root systems. In particular, we show that they form a lattice under containment order and coincide with the initial sections of reflection orders. Using those results, we apply biclosed sets to the study of torsion classes in Calabi-Yau categories. In particular, we show that biclosed sets give rise to generalized stability conditions on the representation category of an affine preprojective algebra and coincide with the restriction of torsion classes to the subcategory of spherical objects. In the case of type $\widetilde{A}$, biclosed sets case admit an explicit combinatorial model which can be interpreted (conjecturally) in terms of homological mirror symmetry; we use this model to parametrize the spherical objects of the representation category. We further use these result to give the first construction of Cambrian lattices for affine-type cluster algebras, giving a Coxeter-theoretic description of the exchange graph of the cluster algebra. We then turn to applications of biclosed sets in the Bruhat order on a Coxeter group, giving a new proof of EL-shellability, a generalization of the Gelfand-Serganova theorem on Coxeter matroids to infinite Coxeter groups, and a description of the faces of Bruhat interval polytopes. We also prove the broadest known case of the combinatorial invariance conjecture for Kazhdan–Lusztig R-polynomials in the symmetric group, and prove a related conjecture of Google DeepMind and Geordie Williamson in the case of lower intervals in the symmetric group.

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Chapter 1

Introduction

A finite *root system* is a finite set of vectors Φ invariant under Euclidean reflection over the members of Φ. Root systems appear throughout mathematics, from Platonic solids to del Pezzo surfaces to finite simple groups and many places in between. Root systems are intimately tied with their *Weyl groups*, which is the group W generated by the reflections over members of Φ.

In this thesis, we will treat several problems related to root systems, Weyl groups, and their applications. Our focus is on *infinite root systems*. It turns out that in many instances where objects come with a finite root system, there are also more exotic objects with an infinite root system. For instance, the blowup of $\mathbb{P}^2$ at k points is associated to the root system E_k. This is a del Pezzo surface when $k \leq 8$, which is exactly when the root system E_k is finite. However, one can also blowup at 9 or more points, at which point the surface is no longer del Pezzo but is still associated to a (infinite) root system. For another (possibly more familiar) example, we could extend the family of Platonic solids (associated to certain finite root systems) to include tessellations of the plane by regular polygons, which are associated to the *affine root systems*. The root system E_9 governing the blowup of the plane at 9 points is also an affine root system; they are the simplest and most prevalent infinite root systems.

We group our main results into three chapters. Chapter 3 addresses Matthew Dyer's conjectures on *biclosed sets* and *extended weak order*. Biclosed sets are a generalization of *positive root systems*, which are divisions of Φ in half. They were introduced to study Kazhdan–Lusztig theory, but have turned out (in part due to the results presented here) to have applications in algebra, geometry, and combinatorics. Dyer's conjectures relate biclosed sets to *reflection orders* and posit that biclosed sets form a *lattice* under inclusion order. In joint work with David Speyer, we resolve these conjectures for the affine root systems. Here we present also strengthenings of these results that will be used in the remaining chapters.

Chapter 4 relates to *Calabi-Yau categories* and *preprojective algebras*. Calabi-Yau categories are triangulated categories that satisfy a form of Poincaré duality, and include coherent sheaves on Calabi-Yau varieties and Fukaya categories of Calabi-Yau manifolds. Preprojective algebras are algebras originating in quiver representation theory, and their representation theory categorifies the geometry of root systems. It turns out that many subcategories of 2-Calabi-Yau categories can be realized as categories of preprojective algebra representations. These are the subcategories generated by *spherical objects*, which include line bundles on exceptional curves of Calabi-Yau surfaces and Lagrangian spheres in Fukaya categories of Calabi-Yau surfaces. We prove that when spherical objects generate an affine root system, then the *tilting* and *stability* theory is governed by the biclosed sets in the root system. We conjecture extensions of this result to more general root systems. Our results imply that biclosed sets also describe *cluster algebras* of affine type.

Chapter 5 relates to *Bruhat order* on a Weyl group W. This is the order on W which describes the inclusion of Schubert varieties in a flag variety, but it is defined for any Coxeter group. We show how to use biclosed sets to give new proofs of some results of Matthew Dyer, including an elementary proof of the existence of EL-labelings on Bruhat intervals. We then use biclosed sets to state and prove a generalization of the Gelfand–Serganova theorem characterizing Coxeter matroids to infinite Coxeter groups, and apply this to prove that faces of Bruhat interval polytopes are Bruhat interval polytopes in an arbitrary Coxeter group. We also describe our new approach to the combinatorial invariance conjecture for Kazhdan–Lusztig polynomials introduced with Christian Gaetz in [5], which proves the most general known result for $W = S_n$, and generalize the main recurrence to other Coxeter groups.

We emphasize a meta-pattern appearing across this thesis. Tilting in a triangulated category changes which objects are in "degree 0". Each object M which is in degree 0 before the tilt has a maximal subobject $\Xi(M) \subseteq M$ still in degree 0 after the tilt. For module categories of a type $\widetilde{A}$ preprojective algebra, we show in Chapter 4 that each choice of *torsion class* (tilting direction) has a corresponding biclosed set of roots. In Chapter 5, we show that certain biclosed sets A define a maximal element P_A in a *Coxeter matroid* P. In special cases, such as *MV polytopes*, both $\Xi(M)$ and P_A make sense and coincide, but their domains of definition each encompass many other examples. This meta-pattern motivated many of the results in Chapter 4 and Chapter 5. In particular, *stability conditions* and *reflection orders* are key tools in the proofs, and map to one another under the correspondence between torsion classes and biclosed sets.

In Chapter 2, we go over preliminaries. This includes some routine items, but also features discussion on the subtleties of infinite root systems, infinite hyperplane arrangements, and profinite lattices. In Chapter 3 we discuss biclosed sets and prove conjectures of Dyer for affine root systems. In Chapter 4 we describe Calabi-Yau categories and prove there is a quotient map from torsion classes to biclosed sets in affine type. In Chapter 5 we give a new proof that Bruhat intervals have EL-labelings, prove a generalization of the Gelfand–Serganova theorem to infinite Coxeter groups, and prove new cases of the combinatorial invariance conjecture for Kazhdan–Lusztig R-polynomials. Proofs that appear elsewhere are omitted; all proofs given in this dissertation are new and do not appear in publications unless otherwise indicated.

1.1 Biclosed sets

Matthew Dyer introduced biclosed subsets of a root system Φ. The motivation was to define *twisted Bruhat orders* [25], which play a key role in Dyer's extension of the *Kazhdan–Lusztig conjecture*, describing the composition factors of Verma modules in category $\mathcal{O}$. A subset $B \subseteq \Phi^+$ is biclosed if for all $\alpha, \beta, \gamma \in \Phi^+$ with $\gamma = a\alpha + b\beta$ and $a, b > 0$, we have $\alpha, \beta \in B$

implies $\gamma \in B$ and $\alpha, \beta \notin B$ implies $\gamma \notin B$. The definition of biclosed sets is convex-geometric, and this is a natural perspective from which to study them, but as we will see they have surprising applications in algebra and geometry.

Given a Coxeter group (W, S), the *extended weak order* of W is the poset $\mathrm{Bic}(W)$ whose elements are biclosed sets of positive roots for W, ordered by containment. (This poset turns out to be independent of the choice of root system for W.) Extended weak order is the subject of many open conjectures [27]. When W is a finite Coxeter group, such as S_n, the extended weak order of W can be identified with the *weak order* on W. For infinite Coxeter groups, the weak order embeds as an order ideal of $\mathrm{Bic}(W)$, but there are more elements of $\mathrm{Bic}(W)$. If W is an affine Weyl group of type $\widetilde{A}, \widetilde{B}, \widetilde{C}$, or $\widetilde{D}$, then there are explicit combinatorial models for $\mathrm{Bic}(W)$, which we introduced in [8]. In particular, if W is the affine symmetric group $\widetilde{S}_n$, which is the Coxeter group of type $\widetilde{A}_{n-1}$, then elements of $\mathrm{Bic}(W)$ can be identified with *translation-invariant total orders* (see Definition 3.4.2).

Dyer has two main conjectures on biclosed sets. The first implies that maximal chains of biclosed sets induce total orderings of Φ^+ called *reflection orders* (see Definition 3.0.4).

Conjecture 1.1.1 (Dyer's Conjecture A)**.** *Let $\mathcal{C}$ be a maximal chain in $\mathrm{Bic}(\Phi^+)$. Then for any $\alpha, \beta \in \Phi^+$, there exists a biclosed set $B \in \mathcal{C}$ so that either $\alpha \in B$ and $\beta \notin B$, or $\alpha \notin B$ and $\beta \in B$.*

A *complete lattice* is a partial order so that any set has a least upper bound and a greatest lower bound. The weak order on a finite Coxeter group is known to be a complete lattice, whereas this is never the case for an infinite Coxeter group. In contrast, Dyer's second conjecture asserts that extended weak order is always a complete lattice.

Conjecture 1.1.2 (Dyer's Conjecture B)**.** *For any Coxeter group W, the extended weak order $\mathrm{Bic}(W)$ is a complete lattice.*

In joint work with David Speyer, we prove these conjectures for affine root systems.

Theorem 1.1.3. *Dyer's Conjectures A and B are true for affine root systems.*

To prove Dyer's conjectures, we define *suitable orderings* on the root system. We conjecture in Conjecture 3.0.7 that such orderings exist for any root system. We show how our methods extend to prove more useful properties of $\mathrm{Bic}(W)$ beyond Dyer's conjectures, which we shall use in the later sections.

Theorem 1.1.4. *If W is an affine Weyl group, then* $\mathrm{Bic}(W)$ *is a profinite semidistributive lattice.*

1.2 Calabi-Yau categories

Let A be a finite-length $\mathbb{C}$-linear abelian category. Then the *lattice of torsion classes* of A is a poset $\mathsf{Tors}(\mathsf{A})$ whose elements are *torsion classes* in A, ordered by containment. A torsion class is a collection of objects closed under isomorphism, quotients, and extensions. This lattice is studied in the context of τ-tilting and silting theory for algebras. In particular, the poset diagram of $\mathsf{Tors}(\mathsf{A})$ includes the τ-tilting exchange graph, which itself generalizes the exchange graph for mutations in a *cluster algebra*. Torsion classes are closely related to *stability conditions* and *t-structures* on a triangulated category.

Each quiver Q has an associated *preprojective algebra* Π_Q over $\mathbb{C}$. When Q is an oriented Dynkin diagram associated to the finite Coxeter group W, then there is an isomorphism of lattices $\mathsf{Tors}(\Pi_Q) \xrightarrow{\sim} \mathrm{Bic}(W)$ [42]. Our main result of this chapter is the analog of this fact when W is an affine Weyl group.

Theorem 1.2.1. *Let Q be an affine quiver with Weyl group W. Then there is a quotient map of complete lattices*

$$\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathrm{Bic}(W).$$

We prove a stronger version, which allows us to extract information about Π_Q-modules from the lattice theory of $\mathrm{Bic}(W)$. A module M for Π_Q is called a *spherical module* (or a *real brick*) if $\mathrm{Ext}^i(M,M) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$. In other words, M has the same cohomology groups as a 2-sphere. Our second main result both classifies the spherical modules of Π_Q in terms of join-irreducible elements of $\mathrm{Bic}(W)$, and also describes the existence of maps between spherical modules using the poset structure of $\mathrm{Bic}(W)$. We write $\mathsf{JIrr}^c(\mathrm{Bic}(W))$ for the *completely join-irreducible elements* of $\mathrm{Bic}(W)$, and we write $\mathsf{Sph}(\Pi_Q)$ for the spherical modules. In type $\widetilde{A}$, we will index elements of both sets by diagrams (denoted below by $\mathfrak{a}$) as described in Sections 3.4 and 4.5.5.

Theorem 1.2.2. *Let Q be an affine quiver with affine Weyl group W. There is a bijection*

$$\mathsf{JIrr}^c(\mathrm{Bic}(W)) \xrightarrow{\sim} \mathsf{Sph}(\Pi_Q)$$
$$J \mapsto \Pi_{\sigma(J)}.$$

Furthermore, the following are equivalent:

- *There exists a non-zero map of Π_Q-modules from $\Pi_{\mathfrak{a}(J_1)}$ to $\Pi_{\mathfrak{a}(J_2)}$;*
- *There does not exist $R \in \mathrm{Bic}(W)$ so that $R \geq J_1$ and J_2 covers $R \wedge J_2$.*

In type $\widetilde{A}$, the objects put in bijection by Theorem 1.2.2 are also in bijection with a third, geometric object called *stability domains* or *shards*. In their recent work [22], Dana, Speyer, and Thomas showed that shards inject into $\mathsf{Sph}(\Pi_Q)$; the resulting real bricks are called *shard modules*. In Corollary 4.5.17, we show that every spherical module of Π_Q is a shard module when Q is of type $\widetilde{A}$. This is known to be false for a general quiver. We will also prove [22, Conjecture 6.11], about the existence of maps between spherical objects, for any quiver.

There are many motivations for studying Π_Q-modules. One reason is their connection to canonical bases for the quantum groups and for coordinate rings of flag varieties. For example, each Π_Q-module has an associated polytope, its *Harder–Narasimhan polytope*. The Harder–Narasimhan polytope of a *generic* Π_Q-module is an *MV polytope*[1], and MV polytopes

[1] In affine type, MV polytopes are tagged with additional data arising from the positive multiplicity of imaginary roots.

give a realization of Kashiwara crystals [9]. Spherical modules are all generic in this sense, so they give rise to MV polytopes. Torsion classes are also an important tool for studying more general MV polytopes.

Another motivation for studying Π_Q-modules is to understand cluster algebras. Each cluster algebra (with a choice of initial seed) gives rise to a directed graph called the *ordered exchange graph*, which describes the mutations connecting different seeds [18]. For finite-type cluster algebras, these graphs are an orientation of the 1-skeleton of an associahedron of the appropriate type.

Theorem 1.2.3. *Let Q be an affine quiver, and let A_Q be the cluster algebra (with principal coefficients) associated to Q. Then there is a lattice congruence $\equiv_Q$ of* $\mathrm{Bic}(W)$ *so that the ordered exchange graph of A_Q embeds as a subgraph of the Hasse diagram of* $\mathrm{Bic}(W)/\equiv_Q$.

We deduce Theorem 4.5.13 essentially formally from Theorem 1.2.2 together with results from cluster-tilting theory. An interesting question is to describe these quotients combinatorially, which would give a new way of describing affine-type cluster algebras. In finite type, this question is answered via the *Cambrian lattices*, which are quotients of weak order on a finite Coxeter group. As a partial answer to the question, in recent work with Colin Defant [3] we describe the combinatorics of $\mathrm{Bic}(W)/\equiv_Q$ in the case where Q is the oriented cycle. The resulting lattice quotient is the *affine Tamari lattice*; it can be identified with the sublattice of $\mathrm{Bic}(W)$ consisting of 312-avoiding translation-invariant total orders, and has a description using translation-invariant binary trees.

We remark on one more motivation for studying Π_Q modules: the McKay correspondence associates to the cycle quiver an *ADE singularity*; in our case, the singularity is the zero locus X of the polynomial $x^2 + y^2 + z^n$ in $\mathbb{C}^3$. There is a minimal resolution of singularities $Y \to X$, and one version of the McKay correspondence asserts that Π_Q-modules are (derived) equivalent to coherent sheaves on the algebraic variety Y [16]. The homological mirror symmetry conjecture predicts that these objects are further (derived) equivalent to certain Lagrangian submanifolds of a symplectic variety [1, Section 9.2]. Spherical modules correspond to the Lagrangian submanifolds which are diffeomorphic to 2-spheres. The arc diagrams we describe in Section 3.4 (to parametrize spherical modules in type $\widetilde{A}$) also arise in symplectic geometry as the image of Lagrangian spheres under a Lefschetz fibration. This particular case of homological mirror symmetry seems to be unproven, but we will use intuition motivated by this picture in our argument. It would be an interesting question to interpret our results on the symplectic geometry side of mirror symmetry.

1.3 Bruhat order

Kazhdan–Lusztig polynomials

Let G be the complex semisimple Lie group (or Kac–Moody group) associated to the root system Φ, with Borel subgroup B. Then G/B is the flag variety for G. Specializing to the case $G = \mathrm{SL}_n$, then G/B is the moduli space of complete flags $\{V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim V_i = i\}$ in $\mathbb{C}^n$. This flag variety can be decomposed into Schubert cells X_v where v is an element of Weyl group W. The closure $\overline{X}_v$ of a Schubert cell in G/B is a *Schubert variety*, which are fundamental in representation theory and algebraic geometry. These are generally singular varieties, so their cohomology groups may not satisfy Poincaré duality. In this case, *intersection cohomology* provides an interesting invariant of $\overline{X}_v$. The stalk of the intersection cohomology complex on $\overline{X}_v$ at a point $x_u \in X_u$ is a complex of vector spaces which is encoded by the *Kazhdan–Lusztig polynomial* $P_{u,v}(q)$ [37]. More precisely, the coefficient of q^i in $P_{u,v}$ is the dimension of $H^{2i}(\mathbf{IC}(\overline{X}_v)_{x_u})$.

Kazhdan–Lusztig polynomials additionally encode the transition matrix for the *canonical basis* of a Hecke algebra [36] and, by the Kazhdan–Lusztig conjecture, the multiplicities of simple Lie algebra modules inside Verma modules and the dimensions of Ext groups between them [19].

The *Bruhat order* on a Weyl group, defined by $u \leq v$ if and only if $X_u \subseteq \overline{X}_v$), plays a crucial role in the computation of Kazhdan–Lusztig polynomials. The Bruhat order is a graded poset, with the rank of v denoted $\ell(v)$ and coinciding with the complex dimension of

$\overline{X}_v$. Although there are many interpretations of $P_{u,v}$, most are computationally inefficient, scaling with $\ell(v)$ and requiring a traversal of group elements much less than v in Bruhat order to compute P_{uv}. A different method introduced by Dyer [26] requires only the group elements in the Bruhat order interval $[u, v]$. However, more than thirty years ago it was conjectured by Lusztig and Dyer that far less information is required to compute P_{uv}. Rather than the group elements in $[u, v]$, the *combinatorial invariance conjecture* (e.g., [15, 44]) states that one needs only the interval $[u, v]$ viewed as an *abstract, unlabeled* poset to compute $P_{u,v}$. More precisely:

Conjecture 1.3.1. *If u, v, u', v' are elements of a Coxeter group, and the Bruhat interval $[u, v]$ is isomorphic as a poset to the Bruhat interval $[u', v']$, then $P_{u,v} = P_{u',v'}$.*

Call the data associated to a pair u, v *combinatorial* if it depends only on the poset isomorphism class of $[u, v]$. The conjecture asserts that $P_{u,v}$ is combinatorial. The polynomials $P_{u,v}$ are computed from another family of polynomials, the *R-polynomials* $R_{u,v}$. If $R_{x,y}$ is combinatorial for all $x, y \in [u, v]$, then $P_{u,v}$ is combinatorial. In joint work with Christian Gaetz we show the following, which generalizes the broadest known case of combinatorial invariance for the symmetric group, the case when $u = u' = e$.

Theorem 1.3.2. *If $[u, v]$ is isomorphic to an interval with linearly independent roots labeling its atoms, then $R_{u,v}$ is combinatorial.*

Recently, a team from Google DeepMind and Geordie Williamson refined Conjecture 1.3.1 [13, 23]. They give a conjectural recurrence which computes $P_{u,v}$ in terms of combinatorial data and the Kazhdan–Lusztig polynomial of strictly smaller (combinatorially specified) intervals; however they did not show that their recurrence works for any infinite family of intervals. Their recurrence uses the *hypercube decompositions* of the interval. Our second main result (joint with Gaetz) is the following.

Theorem 1.3.3. *The DeepMind–Williamson conjecture holds for intervals of the form $[e, v]$.*

Other topics

In this chapter we also address other topics in Bruhat order. For instance, we use biclosed sets to give a new proof of a well-known theorem of Dyer.

Theorem 1.3.4. *Let* $[u, v]$ *be an interval in the Bruhat order of a Coxeter group. Then any reflection order gives an* EL-labeling *of* $[u, v]$*.*

Prior proofs all run a recurrence based on multiplication in Hecke algebras that is quite involved. We also use this result to give a generalization and proof of the *Gelfand–Serganova theorem* [14] for infinite Coxeter groups.

Theorem 1.3.5. *The following are equivalent, for a convex hull* P *of finitely many regular weights in a* W*-orbit containing a dominant weight.*

- *Each edge of* P *is parallel to a real root;*
- *For each* twisted Bruhat order $\leq_w$*, there is a unique* $\leq_w$*-minimal element of* P*.*

We apply this to prove that faces of Bruhat interval polytopes are Bruhat interval polytopes for arbitrary Coxeter groups. For finite Weyl groups, this was first proven using total positivity and canonical bases in [49].

Theorem 1.3.6. *Let* ρ *be a dominant weight and let* $[u, v]$ *be a Bruhat interval. Then any face of the convex hull of* $[u, v]\rho$ *is of the form* $[x, y]\rho$ *for some* $x, y \in [u, v]$*.*

The techniques of this section are motivated by the properties of *MV polytopes*, of which the Bruhat interval polytope of a lower interval in a Weyl group is a special case.

Chapter 2

Background

2.1 Infinite hyperplane arrangements

Let V be a real vector space. A **hyperplane** in V is a codimension 1 subspace $H \subset V$. Given a hyperplane H, a **half-space bounded by** H is a connected component of $V \setminus H$. For each hyperplane H, there are two half-spaces bounded by H. When we are in a context where there is a distinguished "positive side" of H, then we will denote these half-spaces by H^+ and H^-. We have the decomposition $V = H^+ \sqcup H \sqcup H^-$.

A **hyperplane arrangement** is a set of hyperplanes. If $\mathcal{A}$ is a finite hyperplane arrangement, then we can define a region of $\mathcal{A}$ to be a connected component of $V \setminus \bigcup_{H \in \mathcal{A}} H$.

Example 2.1.1. Let $H_{i,j}$ be the hyperplane in $\mathbb{R}^n$ defined by $H_{i,j} = \{\mathbf{x} \in \mathbb{R}^n \mid x_i = x_j\}$. The **braid arrangement** is $\mathcal{B}_n := \{H_{i,j} \mid 1 \leq i < j \leq n\}$. Two points $\mathbf{x}, \mathbf{x}'$ in $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{B}_n} H$ are in the same region if and only if $x_i < x_j$ whenever $x'_i < x'_j$. Equivalently, two points are in the same region if their coordinates are ordered the same way. A 2-dimensional slice through $\mathcal{B}_3$ is depicted in Figure 2.1.

Defining regions for infinite hyperplane arrangements is more subtle; for instance, how would one define a region for the arrangement consisting of all hyperplanes in V? To motivate the definition, consider that in a finite hyperplane arrangement $\mathcal{A}$, any region R is determined by the half-spaces which are bounded by hyperplanes in $\mathcal{A}$ and contain R. See Figure 2.1

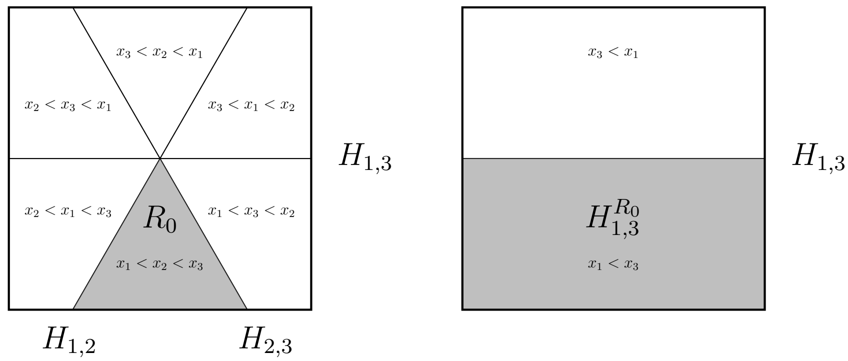


Figure 2.1: The braid arrangement $\mathcal{B}_3$. On the left, a the points of a region R_0 are indicated by shading. On the right, the half-space $H_{1,3}^{R_0}$ is indicated by shading.

for an example. In an infinite hyperplane arrangement, rather than viewing a region as a set of points, we shall instead view it as a choice of half-space bounded by H for each $H \in \mathcal{A}$.

Definition 2.1.2. Let $\mathcal{A}$ be a hyperplane arrangement in a real vector space V. A **pre-region** of $\mathcal{A}$ is a function R from $\mathcal{A}$ to the half-spaces of V, so that the half-space assigned to $H \in \mathcal{A}$ is bounded by H. We write H^R for the half-space assigned to H; we say that H^R **contains** R. The **points** of R in a subarrangement $\mathcal{A}' \subseteq \mathcal{A}$ are the elements of $\bigcap_{H \in \mathcal{A}'} H^R$. We say that R is a **region** if R has points in every finite subarrangement of $\mathcal{A}$. We say that R is a **strict region** if R has points in $\mathcal{A}$.

We refer to the points of a strict region R in $\mathcal{A}$ itself (i.e. elements of $\bigcap_{H \in \mathcal{A}} H^R$) simply as the points of R. Any element of $V \setminus \bigcup_{H \in \mathcal{A}} H$ is a point of a unique strict region. In a finite hyperplane arrangement $\mathcal{A}$, every region is strict and the collection of points of a region R is a connected component of $V \setminus \bigcup_{H \in \mathcal{A}} H$. Hence for finite hyperplane arrangements, our definition of region coincides with the usual one.

Remark 2.1.3. Any hyperplane arrangement $\mathcal{A}$ determines an *oriented matroid* $\mathcal{M}$. We have defined the regions of $\mathcal{A}$ to coincide with the *topes* of $\mathcal{M}$ (a matroid-theoretic abstraction of regions). Meanwhile, strict regions cannot be determined from $\mathcal{M}$; they depend on the *realization* of $\mathcal{M}$.

Example 2.1.4. Let $H_k \subseteq \mathbb{R}^2$ be the hyperplane $H_k = \{(x_1, x_2) \mid x_1 - kx_2 = 0\}$ and $H_\infty = \{(x_1, x_2) \mid x_2 = 0\}$, and let $\mathcal{A} = \{H_k \mid k \in \mathbb{N}\} \cup \{H_\infty\}$. For each $k \in \mathbb{N} \cup \{\infty\}$, let H_k^+ be the

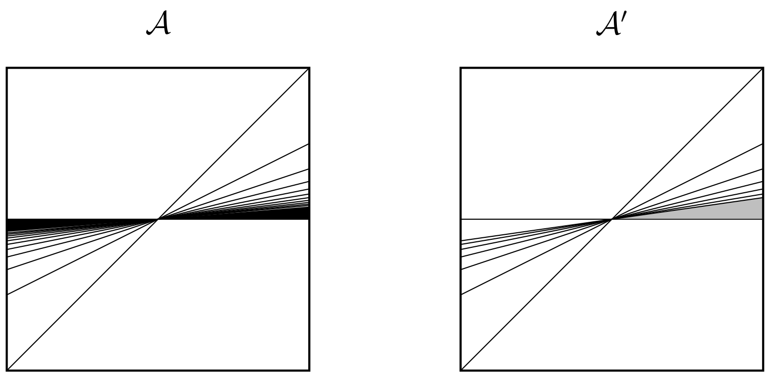


Figure 2.2: The hyperplane arrangement $\mathcal{A}$ from Example 2.1.4. On the right is a finite subarrangement $\mathcal{A}'$, with the points of R in $\mathcal{A}'$ indicated with shading.

half-space bounded by H_k which contains the point $(0,-1)$, and let H_k^- be the half-space bounded by H_k containing the point $(0,1)$. There is a region R with $H_k^R = \begin{cases} H_k^+ & \text{if } k \neq \infty \\ H_k^- & \text{if } k = \infty \end{cases}$. Then R is not a strict region. Neither is $-R$. All other regions of $\mathcal{A}$ are strict.

Let R_1, R_2 be pre-regions. We say that a hyperplane H **separates** R_1 from R_2 if $H^{R_1} \neq H^{R_2}$. The **separating set** $\mathcal{S}(R_1, R_2) \coloneqq \{H \in \mathcal{A} \mid H^{R_1} \neq H^{R_2}\}$ is the collection of hyperplanes separating R_1 from R_2. We say that R_1 and R_2 are **adjacent** if $|\mathcal{S}(R_1, R_2)| = 1$. We say that a hyperplane H **bounds** a region R (or that H is **incident** to R or that H is a **wall** of R) if there is a region R' so that $\mathcal{S}(R, R') = \{H\}$. Unlike the finite case, in an infinite hyperplane arrangement there may be regions with no bounding hyperplanes.

Example 2.1.5. Let $\mathcal{A}$ be the collection of lines in $\mathbb{R}^2$ with rational slope (along with the vertical line). Then the strict regions of $\mathcal{A}$ correspond to rays of irrational slope. Every strict region has no bounding hyperplanes.

Example 2.1.6. Let $\mathcal{A} = \{H \mid H \subseteq V \text{ is a hyperplane}\}$ be the arrangement of all hyperplanes in V. Then $\mathcal{A}$ has no strict regions, but many regions. For example, if $V = \mathbb{R}^2$, then each hyperplane bounds 4 regions of $\mathcal{A}$. Furthermore, every region is bounded by a unique hyperplane.

2.1.1 Points of regions

While some regions may not have points, this turns out to be problem analogous to a polynomial not having roots in $\mathbb{R}$. In particular, it can be fixed by considering a field extension. Let K be an ordered field containing $\mathbb{R}$. For instance, we could take $K = \mathbb{R}(\varepsilon)$ or $K = \mathbb{R}((\varepsilon))$, where ε is a formal variable which is positive and smaller than all positive real numbers. Any hyperplane H in V determines a codimension-1 subspace $H \otimes_{\mathbb{R}} K$ of $V \otimes_{\mathbb{R}} K$, and a half-space $H^{\pm}$ bounded by H determines a half-space of $V \otimes_{\mathbb{R}} K$ bounded by $H \otimes_{\mathbb{R}} K$, denoted $H^{\pm} \otimes_{\mathbb{R}} K$. A **point of R over K** is an element of $\bigcap_{H\in\mathcal{A}}(H^R \otimes_{\mathbb{R}} K)$. By the **line segment** (over K) connecting $p, q \in V \otimes_{\mathbb{R}} K$, we mean the set $\{\lambda_1 p + \lambda_2 q \mid \lambda_1, \lambda_2 \in K,\ \lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1\}$.

Lemma 2.1.7. *Let $\mathcal{A}$ be a hyperplane arrangement in a real vector space V. Let K be an ordered field properly containing $\mathbb{R}$. Then any pre-region with points over K is a region. If additionally V is finite-dimensional, then any region of $\mathcal{A}$ has points over K.*

Proof. Let R be a pre-region of $\mathcal{A}$ and let θ be a point of R over K. Let $\mathcal{A}'$ be a finite subarrangement of $\mathcal{A}$. If $|\mathcal{A}'| = 1$ then R has points in $\mathcal{A}'$. Otherwise, let $H \in \mathcal{A}'$. By induction, R has a real point θ' in the subarrangement $\mathcal{A}' \setminus \{H\}$. Consider the line segment connecting θ and θ'. Because θ and θ' are both points of R over K in $\mathcal{A}' \setminus \{H\}$, the line segment connecting them either intersects no hyperplanes of $\mathcal{A}'$ or intersects H. In the first case, θ' is a point of R in $\mathcal{A}'$. In the second case, let $R_{\theta'}$ be the region of $\mathcal{A}'$ with the point θ'. Then H bounds $R_{\theta'}$, so there is a region R' of $\mathcal{A}'$ so that $\mathcal{S}_{\mathcal{A}'}(R_{\theta'}, R') = \{H\}$. In particular, R' has points and $H^{R'} = H^R$ for all $H \in \mathcal{A}'$, so R has points in $\mathcal{A}'$. Hence R is a region.

For the converse statement, assume V is finite dimensional. Let R be a region of $\mathcal{A}$. Consider the space $\mathbb{P}_{\geq 0}(V) = \{\mathbb{R}_{\geq 0} v \mid v \in V \setminus \{0\}\}$, which is homeomorphic to the sphere $S^{\dim V - 1}$. For each $H \in \mathcal{A}$, let Z_H be the closed subset of $\mathbb{P}_{\geq 0}(V)$ consisting of rays in V that are contained in $H \cup H^R$. Then Z_H has the finite intersection property, because R has points in every finite subarrangement of $\mathcal{A}$. Since $\mathbb{P}_{\geq 0}(V)$ is compact, there must be some element $\mathbb{R}\theta \in \bigcap_{H\in\mathcal{A}} Z_H$. Let $\mathcal{A}' = \{H \in \mathcal{A} \mid \theta \in H\}$. Then θ is a point of R in the subarrangement $\mathcal{A} \setminus \mathcal{A}'$. Since θ is contained in every hyperplane of $\mathcal{A}'$, we can project $\mathcal{A}'$ onto the hyperplane arrangement $\mathcal{A}'/\mathbb{R}\theta := \{H/\mathbb{R}\theta \mid H \in \mathcal{A}'\}$ in $V/\mathbb{R}\theta$, and this induces a bijection between the regions of $\mathcal{A}'$ and $\mathcal{A}'/\mathbb{R}\theta$. By induction on the dimension of V, there is a point $\overline{\alpha}$ of $R/\mathbb{R}\theta$ over K in $\mathcal{A}'/\mathbb{R}\theta$. Any representative $\alpha \in V \otimes_{\mathbb{R}} K$ of $\overline{\alpha}$ is a point of R over K in $\mathcal{A}'$. Fix a basis for V and let $\varepsilon \in K$ be some number so that the coordinates of $\varepsilon\alpha$ are lesser in magnitude than any positive real number.

We claim that $\theta + \varepsilon\alpha$ is a point of R over K. Indeed, if H is in $\mathcal{A}'$ then $\theta + \varepsilon\alpha$ is in the same half-space bounded by H as α, which is H^R. If instead H is in $\mathcal{A} \setminus \mathcal{A}'$ then we claim that the line segment connecting θ and $\theta + \varepsilon\alpha$ does not intersect H. Indeed, there is a positive real distance from θ to H. If we compute distance using the Euclidean norm with respect to our chosen basis for V, then we see by choice of ε that the distance from θ to $\theta + \varepsilon\alpha$ is less than any positive real number. Hence the line segment does not intersect H, so $\theta + \varepsilon\alpha$ is in the same half-space as θ, which is H^R. It follows that $\theta + \varepsilon\alpha$ is a point of R over K. □

Let R be a pre-region of the hyperplane arrangement $\mathcal{A}$. If $H \in \mathcal{A}$, then the **restriction** of $\mathcal{A}$ to H is the hyperplane arrangement $\mathcal{A}^H := \{H \cap H' \mid H' \in \mathcal{A},\ H' \neq H\}$.

Lemma 2.1.8. *Let R be a region of $\mathcal{A}$ and $H \in \mathcal{A}$. If $H \in \mathcal{A}$ is a bounding hyperplane of R, then there is a unique region of $\mathcal{A}^H$, denoted $R \cap H$, so that $(R \cap H)^{H' \cap H} = R^{H'} \cap H$ for all $H' \in \mathcal{A}$ distinct from H.*

Proof. We first need to check that if $H_1', H_2' \in \mathcal{A}$ satisfy $H_1' \cap H = H_2' \cap H$, then $R^{H_1'} \cap H = R^{H_2'} \cap H$. Indeed, otherwise we would have $R^{H_1'} \cap R^{H_2'} \cap H = \varnothing$, but this contradicts the fact that R is bounded by H. Hence $R \cap H$ is well-defined. To see that it is a region, let $\mathcal{A}' \subseteq \mathcal{A}$ be a finite subarrangement of hyperplanes distinct from H. Then $\bigcap_{H' \in \mathcal{A}' \cup \{H\}} R^{H'} \neq \varnothing$, so $\bigcap_{H' \in \mathcal{A}'} (H \cap R)^{H' \cap H} \neq \varnothing$. □

Proposition 2.1.9. *Assume V is finite dimensional. Let $\mathcal{A}$ be a hyperplane arrangement and $H \in \mathcal{A}$. Then the map $R \mapsto R \cap H$ is a surjective, 2-1 mapping from regions of $\mathcal{A}$ with bounding hyperplane H to regions of $\mathcal{A}^H$.*

Proof. Let R' be a region of $\mathcal{A}^H$. There are at most two regions R of $\mathcal{A}$ so that H is a wall of R and $R \cap H = R'$, since for such a region the values of ${H'}^R$ are determined for every $H' \neq H$. We will show there are exactly two such regions.

Let K be a non-archimedean ordered field containing $\mathbb{R}$ and let p be a point of R' over K. Let p' be some element of $V \setminus \{H\}$. Then there exists some $\varepsilon > 0$ in K so that the line segment from $p - \varepsilon p'$ to $p + \varepsilon p'$ does not intersect any hyperplane besides H. (One may take any positive ε which is much smaller than the coordinates of p and is smaller than any real number.) Then $p + \varepsilon p'$ is a point of some pre-region R_1, which is a region by Lemma 2.1.7. Similarly there is a region R_2 of which $p - \varepsilon p'$ is a point over K. We have $\mathcal{S}(R_1, R_2) = \{H\}$, so there are two regions R of $\mathcal{A}$ so that $R \cap H = H$. $\square$

Definition 2.1.10. A region R is **regular** if, for every region $R' \neq R$, there is a hyperplane $H \in \mathcal{S}(R, R')$ which bounds R. A **based hyperplane arrangement** is a hyperplane arrangement $\mathcal{A}$ equipped with a strict regular region R_0, its **base region**.

We note that, in general, strict does not imply regular, and regular does not imply strict.

Example 2.1.11. We continue with the hyperplane arrangement $\mathcal{A}$ from Example 2.1.4. Every strict region of $\mathcal{A}$ is regular. The non-strict region R is not regular, since if we let $-R_0$ be the region containing $(0, 1)$, then $\mathcal{S}(R, -R_0)$ does not contain H_∞, the unique bounding hyperplane of R.

Remark 2.1.12. We have defined a based hyperplane arrangement to have a strict regular region as its base region. There is essentially no difference if we drop the requirement of strictness. However, in every example we will consider, every regular region will also be strict. In part, this is due to the following result.

Proposition 2.1.13. *Let R be a regular region of a hyperplane arrangement $\mathcal{A}$ in a real vector space V. If either V is finite dimensional or R has finitely many walls, then R is strict.*

Proof. We can assume $\mathcal{A}$ is an essential arrangement (meaning that $\bigcap_{H \in \mathcal{A}} H = \{0\}$). It follows then under either hypothesis that V is finite dimensional. Let R be a regular region of $\mathcal{A}$. If R has at most one wall, then $\dim V \leq 1$ and the claim is trivial. Otherwise, there are distinct walls H_1, H_2 of R. Consider the region $R \cap H_1$ of $\mathcal{A}^{H_1}$. It is a regular region, since any other region of $\mathcal{A}^{H_1}$ is of the form $R' \cap H_1$ for a unique region R' of $\mathcal{A}$ so that ${R'}^{H_1} = R^{H_1}$ by Proposition 2.1.9. It follows by induction on the dimension of V that $R \cap H_1$ has a point p_1 in $\mathcal{A}^{H_1}$. Similarly $R \cap H_2$ has a point p_2 in $\mathcal{A}^{H_2}$. Then the midpoint of p_1 and p_2 is a point of R, so R is strict. $\square$

2.2 Reflection groups and Coxeter groups

If x is an element of a group, then we write $|x|$ for the order of x. The notation $\langle S \mid R \rangle$ denotes the presentation of a group by generators S and relations R.

Definition 2.2.1. Let (W, S) be a pair consisting of a group W and a generating subset $S \subseteq W$ of order 2 elements. Then (W, S) is a **Coxeter system** if the canonical map

$$\langle S \mid (st)^{|st|} = 1,\ s, t \in S \rangle \to W$$

is an isomorphism.

In other words, (W, S) is a Coxeter system if all relations among the elements of S can be deduced from relations of the form $(st)^k = 1$. When there is an understood generating set S so that (W, S) is a Coxeter system, then we say that W is a **Coxeter group**. Elements of S are called the **simple reflections** of W. The size $|S|$ is the **rank** of W. An isomorphism of Coxeter groups is an isomorphism of groups which sends simple reflections to simple reflections.

Let $S = \{s_i\}_{i \in I}$ be the set of simple reflections of a Coxeter group W. Then W is determined up to isomorphism by the matrix encoding the orders $m_{ij} = |s_i s_j|$ for $i, j \in I$. This is called the **Coxeter matrix** of W. As shorthand, Coxeter matrices are encoded in **Coxeter diagrams**, which are graphs with vertices labeled by elements of I and an edge between i and j if s_i, s_j do not commute. The edge between i and j is labeled by m_{ij}; this

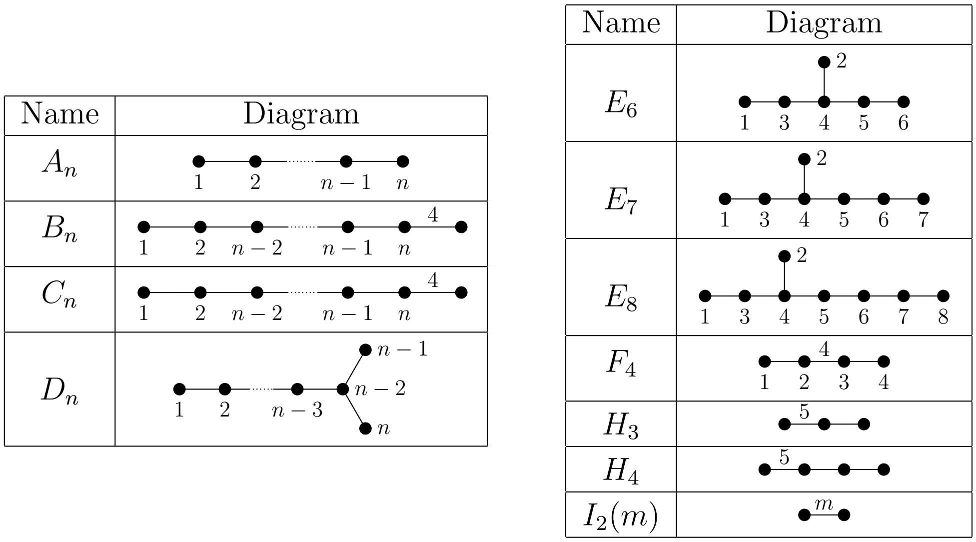


Figure 2.3: The Coxeter diagrams associated to finite Coxeter groups.

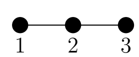
label is typically not shown if $m_{ij} = 3$.

Example 2.2.2. The symmetric group S_4 is a Coxeter group with generating set

$$S = \{(1,2),\ (2,3),\ (3,4)\}.$$

The Coxeter matrix and Coxeter diagram are

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$



We say a Coxeter group is **reducible** if there is a partition $S = S_1 \sqcup S_2$ into nonempty parts such that if $s_1 \in S_1$ and $s_2 \in S_2$ then s_1 and s_2 commute. The following is a theorem of H. S. M. Coxeter.

Proposition 2.2.3. *A Coxeter group W is finite and irreducible if and only if the Coxeter diagram of W is among those shown in Figure 2.3.*

2.2.1 Coxeter groups as reflection groups

Coxeter groups are distinguished by the existence of special representations as reflection groups. It is for this reason that elements of S are called simple reflections. Additionally, given a Coxeter system (W, S) then we define the set $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$. Elements of T are called the **reflections** of W.

Let V be a real vector space with dual vector space V^* and pairing $\langle -, - \rangle : V^* \otimes V \to \mathbb{R}$. The data of a hyperplane H in V is equivalent to the data of the one-dimensional subspace

$$H^{\perp} = \{f \in V^* \mid \forall x \in H,\ f(x) = 0\} \subseteq V^*.$$

A linear map $T \in \mathrm{GL}(V)$ is called a **reflection** if $T^2 = 1$ and the fixed point set $\mathrm{Fix}(T)$ is a hyperplane. Any reflection T is determined uniquely by the hyperplane $\mathrm{Fix}(T)$ and the -1-eigenspace of T, which is a one-dimensional subspace of V. We will often present a reflection T via the data of a generator α of the -1-eigenspace of T, called a **root**, and a generator $\alpha^\vee$ of $\mathrm{Fix}(T)^{\perp}$, called a **coroot**. We always normalize these choices so that $\langle \alpha^\vee, \alpha \rangle = 2$. The unique reflection with root α and coroot $\alpha^\vee$ is the linear map

$$x \mapsto x - \langle \alpha^\vee, \alpha \rangle \alpha.$$

Assume now that V is equipped with a symmetric bilinear form $(-, -) : V \otimes V \to \mathbb{R}$. Then an **orthogonal reflection** is a reflection T so that $(Tx, Ty) = (x, y)$ for all $x, y \in V$. If $\alpha \in V$ satisfies $(\alpha, \alpha) \neq 0$, then there is a unique orthogonal reflection T so that $T\alpha = -\alpha$. We write T_α for this reflection, called the (orthogonal) **reflection over** α. Explicitly,

$$T_\alpha(x) = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha.$$

Then α is a root for T_α. If $(-, -)$ is non-degenerate, then we can use it identify V and V^*, and under this identification the coroot associated to α is $\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha$.

Example 2.2.4. Consider $\mathbb{R}^n$ equipped with the usual dot product. Set $\alpha_{i,j} \coloneqq \mathbf{e}_i - \mathbf{e}_j$. Then $(\alpha_{i,j}, \alpha_{i,j}) = 2$, and the reflection $T_{\alpha_{i,j}}$ acts on a vector by swapping its ith and jth coordinates. The reflection $T_{\mathbf{e}_i}$ acts by negating the ith coordinate.

A **reflection group** is a subgroup of $\mathrm{GL}(V)$ generated by reflections. Let W be a reflection group and $S \subseteq W$ a generating set of reflections. The **reflection arrangement** $\mathcal{A}_{(W,S)}$ is the hyperplane arrangement

$$\mathcal{A}_{(W,S)} \coloneqq \{\mathrm{Fix}(T) \mid \exists w \in W,\ wTw^{-1} \in S\}.$$

We say that $\mathcal{A}_{(W,S)}$ is a **based** reflection arrangement if it is equipped with a strict regular region R_0 so that the bounding hyperplanes of R_0 are exactly the hyperplanes $\{\mathrm{Fix}(T) \mid T \in S\}$. The following is a useful characterization of Coxeter groups. We will prove its dual statement using *root systems* in Proposition 2.3.6.

Proposition 2.2.5. *Let W be a group and $S \subseteq W$ a generating set. Then (W, S) is a Coxeter system if and only if there is a real vector space V and a representation $\rho : W \to \mathrm{GL}(V)$ (necessarily faithful) so that $|\rho(s)\rho(t)| = |st|$ for all $s, t \in S$, ρ sends simple reflections to reflections, and so that $\mathcal{A}_{(\rho(W),\rho(S))}$ is a based reflection arrangement.*

To describe the reflection representations of a Coxeter group arising in this way, it will be convenient to pick a root and coroot for each element of S. Furthermore, this choice will give a collection of normal vectors to the hyperplanes in $\mathcal{A}_{(W,S)}$, which is convenient for computation and exposition. This is one motivation for introducing root systems. As we shall see, root systems also have special significance in connecting W with other algebraic structures.

2.3 Root systems

Let V be a real vector space with dual space V^*. Recall that if we are given $\alpha \in V$ and $\alpha^\vee \in V^*$ so that $\langle \alpha^\vee, \alpha\rangle = 2$, then $T_\alpha : V \to V$ defined by $T_\alpha(x) = x - \langle \alpha^\vee, x\rangle\alpha$ is a reflection.

We say that a tuple $(V, \Pi, V^*, \Pi^\vee)$ is a **root pre-datum** (over $\mathbb{R}$) if $\Pi = \{\alpha_i\}_{i \in I} \subseteq V$ and $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I} \subseteq V^*$ are so that $\langle \alpha_i^\vee, \alpha_i \rangle = 2$ for all $i \in I$. Then we define $T_{\alpha_i} \in \mathrm{GL}(V)$ to be the reflection with root α_i and coroot $\alpha_i^\vee$. Given a set of vectors $X \subseteq V$, we write $\mathrm{cone}(X)$ for the set of vectors of the form $a_1 x_1 + \cdots + a_k x_k$ with each $a_i \in \mathbb{R}_{\geq 0}$ and $x_i \in X$.

Definition 2.3.1. Let $(V, \Pi, V^*, \Pi^\vee)$ be a root pre-datum. Let W be the subgroup of $\mathrm{GL}(V)$ generated by $S = \{T_{\alpha_i}\}_{i \in I}$. Set $\Phi \coloneqq \{w\alpha_i \mid w \in W,\ i \in I\}$. Then $(V, \Pi, V^*, \Pi^\vee)$ is a (reduced) **root datum** if the following hold:

- For all $\alpha \in \Phi$, either $\alpha \in \mathrm{cone}(\Pi)$ or $-\alpha \in \mathrm{cone}(\Pi)$ but not both, and
- For all $i \in I$, we have $\alpha_i \notin \mathrm{cone}(\Pi \setminus \{\alpha_i\})$.

In this case Φ is called a (real) **root system** with **base** Π. Elements of Φ are **roots** and elements of Π are **simple roots**. A root α is **positive** if $\alpha \in \mathrm{cone}(\Pi)$ and **negative** if $-\alpha \in \mathrm{cone}(\Pi)$. If $\langle \alpha_i^\vee, \alpha_j \rangle \in \mathbb{Z}$ for all i, j, then we say Φ is **integral**. If there is a bilinear form $(-,-)$ on V so that $(\alpha_i, \alpha_i) > 0$ for all $i \in I$ and elements of S are orthogonal reflections, then Φ is **symmetrizable**. We say Φ is **crystallographic** if it is symmetrizable and integral. We say a root datum $(V, \Pi, V^*, \Pi^\vee)$ is **dualizable** if $(V^*, \Pi^\vee, V, \Pi)$ is also a root datum. We say a root datum is **separable** if there exists a functional $\rho^\vee \in V^*$ so that $\langle \rho^\vee, \Pi \rangle > 0$ and **simplicial** if Π is linearly independent.

We let $\Phi^+ \coloneqq \Phi \cap \mathrm{cone}(\Pi)$ be the set of positive roots, and similarly let $\Phi^- \coloneqq \Phi \cap \mathrm{cone}(-\Pi)$ be the set of negative roots. By definition of a root system, $\Phi = \Phi^+ \sqcup \Phi^-$. The size $|\Pi|$ is called the **rank** of Φ, while $\dim \mathrm{span}(\Phi)$ is called the **linear rank** of Φ.

Remark 2.3.2. We allow the elements of Π and $\Pi^\vee$ to be linearly dependent, so that rank and linear rank may differ. This is necessary for root subsystems to themselves be root systems, as we will see.

When Φ is symmetrizable, then there is a unique orthogonal reflection T_α over each root $\alpha \in \Phi$, as discussed in Section 2.2. In this case knowledge of $\Pi \subseteq V$ (and the bilinear form on V) is enough to determine Φ and W.

Example 2.3.3. Let $V = \mathbb{R}^n$ equipped with its usual inner product. Recall $\alpha_{i,j} = \mathbf{e}_i - \mathbf{e}_j$. Set $\alpha_i \coloneqq \alpha_{i,i+1}$. Then there is a unique symmetrizable root system with base $\Pi = \{\alpha_i\}_{i\in[n-1]}$. Identifying V^* with $\mathbb{R}^n$, the coroots are $\alpha_i^\vee = \alpha_i$. The group W generated by $S = \{T_{\alpha_i}\}_{i\in I}$ is isomorphic to S_n. The root system is $\Phi = \{\alpha_{i,j} \mid 1 \leq i,j < n,\ i \neq j\}$. It is crystallographic and reduced. Φ is called the root system of type A_{n-1}.

It turns out that root data have a very explicit characterization. The **Cartan matrix** of a root pre-datum is the matrix $(A_{ij})_{i,j\in I}$ with $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

Proposition 2.3.4. *A root pre-datum* $(V, \Pi, V^*, \Pi^\vee)$ *is a root datum if and only if* $(-\Pi) \cap \operatorname{cone}(\Pi) = \varnothing$ *and the following hold, for all* $i \neq j$*:*

(a) $A_{ij} \leq 0$*, and*

(b) $A_{ij} = 0$ *if and only if* $A_{ji} = 0$*, and*

(c) The real number $A_{ij}A_{ji}$ *is in the set* $\{4\cos^2(\frac{\pi}{m}) \mid m \in \mathbb{Z}_{\geq 2}\} \cup [4, \infty)$.

Proof. Let $(V, \Pi, V^*, \Pi^\vee)$ be a root datum. Then for any $i, j \in I$ with $i \neq j$, the tuple $(V, \{\alpha_i, \alpha_j\}, V^*, \{\alpha_i^\vee, \alpha_j^\vee\})$ is also a root datum because $\operatorname{span}(\alpha_i, \alpha_j) \cap \operatorname{cone}(\Pi) = \operatorname{cone}(\alpha_i, \alpha_j)$. Furthermore, in this case α_i and α_j are linearly independent and the root datum is symmetrizable. Hence by [29, Lemma 4.1], the Cartan matrix satisfies properties (a-c).

Conversely, let $(V, \Pi, V^*, \Pi^\vee)$ be a root pre-datum with $(-\Pi) \cap \operatorname{cone}(\Pi) = \varnothing$ and with Cartan matrix $(A_{ij})_{i,j\in I}$ satisfying properties (a-c). Let V' be a vector space with basis $\{\beta_i\}_{i\in I}$, and pick vectors $\{\beta_i^\vee\}_{i\in I} \subseteq V'^*$ so that $\langle \beta_i^\vee, \beta_j \rangle = A_{ij}$ for all $i, j \in I$. By [29, Lemma 4.1], the tuple $(V', \{\beta_i\}_{i\in I}, V'^*, \{\beta_i^\vee\}_{i\in I})$ is a root datum. Let Φ' be its root system. Let $\pi : V' \to V$ be the linear map sending $\beta_i \mapsto \alpha_i$ for each $i \in I$. Then π commutes with the action of W, so it restricts to a surjective map $\pi : \Phi' \to \Phi$. Now let $\beta \in \Phi'$. If $\beta \in \Phi'^+$, then $\pi(\beta) \in \Phi^+$, and if $\beta \in \Phi'^-$ then $\pi(\beta) \in \Phi^-$. Since $(-\Pi) \cap \operatorname{cone}(\Pi) = \varnothing$ implies that $(-\operatorname{cone}(\Pi)) \cap \operatorname{cone}(\Pi) = \{0\}$, it follows that every element of Φ is either in $\operatorname{cone}(\Pi)$ or $-\operatorname{cone}(\Pi)$, but not both.

Now assume to the contrary that $\alpha_i \in \operatorname{cone}(\Pi \setminus \{\alpha_i\})$ for some $i \in I$. Then

$$T_{\alpha_i}\alpha_i = -\alpha_i \in \operatorname{cone}(T_{\alpha_i}(\Pi \setminus \{\alpha_i\})) \subseteq \operatorname{cone}(\Pi \setminus \{\alpha_i\}),$$

where the last containment uses property (a) of the Cartan matrix. This implies that both α_i and $-\alpha_i$ are in cone(Π), a contradiction. Hence $(V, \Pi, V^*, \Pi^\vee)$ is a root datum. $\square$

For many common root systems, the Cartan matrix of Φ can be encoded using a *Dynkin diagram.* The number of nodes in the Dynkin diagram is the rank of Φ. The node labeled i in the Dynkin diagram corresponds to the **simple root** $\alpha_i \in \Pi$. A single edge between nodes i and j indicates that the Cartan matrix entries A_{ij} and A_{ji} are both -1. A double edge with an arrow pointing from i to j indicates that $A_{ij} = -1$ and $A_{ji} = -2$. A double edge with arrows pointing in both directions indicates that $A_{ij} = -2$ and $A_{ji} = -2$. A triple edge with an arrow pointing from i to j indicates that $A_{ij} = -1$ and $A_{ji} = -3$. (In particular, if Φ is symmetrizable then arrows always point from longer roots to shorter roots.) If there is no edge between i and j, then $A_{ij} = A_{ji} = 0$. A root system Φ is **reducible** if there is a partition $I = I_1 \sqcup I_2$ into nonempty parts so that $\langle \alpha_i^\vee, \alpha_j \rangle = 0$ for all $i \in I_1$ and $j \in I_2$; equivalently, if its Dynkin diagram is disconnected.

Proposition 2.3.5. *An irreducible crystallographic root system Φ is finite if and only if its Dynkin diagram is among those in Figure 2.4. If Φ is such a root system, with base $\Pi = \{\alpha_i\}_{i \in I}$, then there is a unique root $\theta \in \Phi^+$ so that $\theta + \alpha_i \notin \Phi$ for all $i \in I$.*

The root singled out in the proposition is called the **highest root** of Φ.

If Φ is a root system with base Π, then the group $W = \langle T_{\alpha_i} \mid i \in I \rangle$ is called the **Weyl group** of Φ. The following is [29, Lemma 4.3], and is dual to Proposition 2.2.5.

Proposition 2.3.6. *Let Φ be a root system with base $\Pi = \{\alpha_i\}_{i \in I}$. Let W be the Weyl group of Φ and let $S = \{T_{\alpha_i}\}_{i \in I}$. Then (W, S) is a Coxeter system.*

If Φ is a root system with Weyl group W, then the Coxeter matrix $(m_{ij})_{i,j \in I}$ of W can

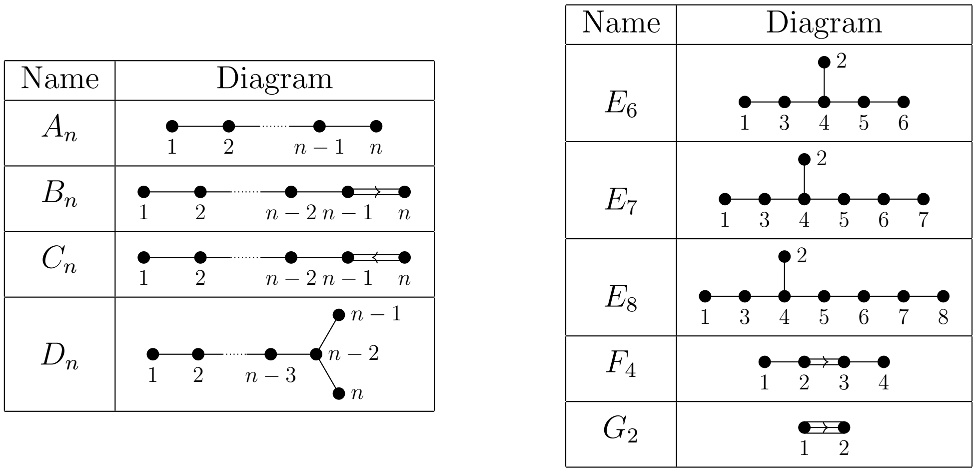


Figure 2.4: The Dynkin diagrams associated to finite crystallographic root systems.

be read off from the Cartan matrix $(A_{ij})_{i,j\in I}$ of Φ. We have, for $i \neq j$,

$$m_{ij} = \begin{cases} m & \text{if } A_{ij}A_{ji} = 4\cos^2 \frac{\pi}{m} \\ \infty & \text{if } A_{ij}A_{ji} \geq 4 \end{cases}.$$

2.3.1 Affine root systems

Let $(V, \Pi, V^*, \Pi^\vee)$ be a root datum defining a finite irreducible crystallographic root system Φ, with highest root θ. Define $\widetilde{V} \coloneqq V \oplus \mathbb{R}\delta$, where δ is a formal symbol. Extend each coroot $\alpha_i^\vee \in V^*$ to a function on $\widetilde{V}$ by declaring $\langle \alpha_i^\vee, \delta\rangle = 0$. We define $\alpha_0 \coloneqq \delta - \theta$ and $\alpha_0^\vee = \theta^\vee$. Then the **affinization** of Φ is the root system $\widetilde{\Phi}$ with root datum $(\widetilde{V}, \Pi \cup \{\alpha_0\}, \widetilde{V}^*, \Pi^\vee \cup \{\alpha_0^\vee\})$. Explicitly, the roots of $\widetilde{\Phi}$ are

$$\widetilde{\Phi} = \{\alpha + k\delta \mid \alpha \in \Phi,\ k \in \mathbb{Z}\}.$$

Root systems arising in this way are called **untwisted affine root systems**. Their Dynkin diagrams are shown in Figure 2.5.

We may also define the affinization of reducible finite crystallographic root systems. If $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_r$ is the decomposition of Φ into irreducible components with highest roots

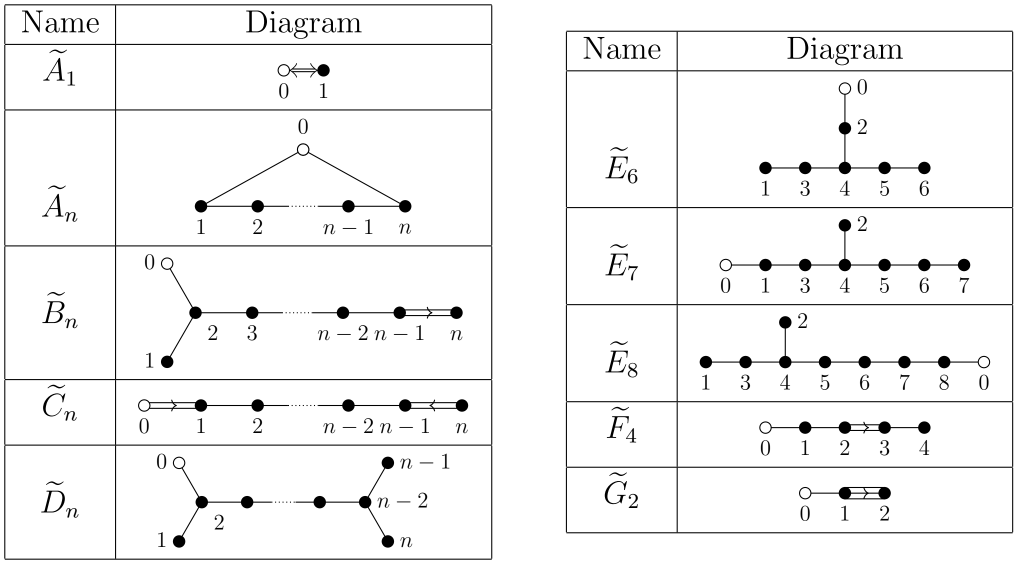


Figure 2.5: The Dynkin diagrams associated to untwisted affine root systems. The node corresponding to α_0 is shown in white.

$\theta_1, \ldots, \theta_r$, then the affinization has root datum

$$(\widetilde{V}, \Pi \sqcup \{\delta - \theta_1, \ldots, \delta - \theta_r\}, \widetilde{V}^*, \Pi^\vee \sqcup \{\theta_1^\vee, \ldots, \theta_r^\vee\})$$

and its roots are $\widetilde{\Phi} = \{\alpha + k\delta \mid \alpha \in \Phi,\ k \in \mathbb{Z}\}$, as above. In particular, if Φ is reducible then the simple roots of $\widetilde{\Phi}$ are linearly dependent.

Example 2.3.7. Let Φ be the root system of type A_{n-1}. Then the highest root θ is $\alpha_{1,n}$. Hence $\alpha_0 = \delta - \alpha_{1,n}$ and $\alpha_0^\vee = \alpha_{1,n}^\vee = \alpha_{1,n}$. Then $\widetilde{\Phi}$ is called the type $\widetilde{A}_{n-1}$ root system. For convenience, we treat the index of α_i to be valued in $\mathbb{Z}/n\mathbb{Z}$, so that $\alpha_i = \alpha_{i+n}$. Then for integers i, j, define

$$\alpha_{i,j} \coloneqq \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}.$$

The roots of $\widetilde{\Phi}$ are

$$\widetilde{\Phi} = \{\alpha_{i,j} \mid i, j \in \mathbb{Z},\ i \not\equiv j \bmod n\}.$$

The positive roots are

$$\widetilde{\Phi}^+ = \{\alpha_{i,j} \in \widetilde{\Phi} \mid i < j\}.$$

2.3.2 Coxeter groups and root systems

Let $(V, \Pi, V^*, \Pi^\vee)$ be a root datum with root system Φ and let W be the Weyl group of Φ, viewed as a Coxeter group with simple generators $S = \{T_\alpha\}_{\alpha \in \Pi}$. By Proposition 2.3.6, each Coxeter group has at least one root system of which it is the Weyl group. Hence we may use root systems to define invariants of W; these invariants may or may not depend on the choice of root system (also called a **realization**) for W. Recall that $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$ denotes the *reflections* of W.

Lemma 2.3.8. *For each root $\alpha \in \Phi$, there is a unique reflection $t_\alpha \in T$ so that $t_\alpha \alpha = -\alpha$. The map $\alpha \mapsto t_\alpha$ restricts to a bijection $\Phi^+ \xrightarrow{\sim} T$. The inverse map sends a reflection t to the unique positive root α_t so that $t\alpha_t = -\alpha_t$.*

Definition 2.3.9. The **length** of an element $w \in W$, denoted $\ell(w)$, is the minimal value of k so that there exists an expression $s_1 \cdots s_k = w$, where $s_i \in S$ for all $1 \leq i \leq k$. An expression $s_1 \cdots s_k = w$ where $k = \ell(w)$ is a **reduced expression** for w. An **inversion** of w is a positive root α so that $w^{-1}\alpha$ is a negative root. The **inversion set** of w is $N(w) \coloneqq \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^-\}$.

Proposition 2.3.10. *Let $w \in W$ and fix a reduced expression $s_1 \cdots s_{\ell(w)} = w$. Then the following are equivalent, for a reflection $t \in T$:*

(a) α_t is an inversion of w;

(b) $\ell(tw) < \ell(w)$;

(c) There exists a (unique) k so that $t = s_1 \cdots s_{k-1} s_k s_{k-1} \cdots s_1$.

In particular we have $\ell(w) = |N(w)|$, and $N(w)$ determines w uniquely.

2.3.3 Root subsystems

Let $\Phi \subset V$ be a root system with Weyl group W. Assume Φ is symmetrizable, so that there is a unique reflection t_α so that $t_\alpha \alpha = -\alpha$, for each $\alpha \in \Phi$. We call a subset Φ' of Φ a **root**

subsystem if, for any roots α and β in Φ', the reflected root $t_\alpha\beta$ is also in Φ'. We write Λ^+ for $\Lambda \cap \Phi^+$. A root subsystem F of Φ is called **full** if for any α and β in F, we have $F \cap \operatorname{span}\{\alpha, \beta\} = \Phi \cap \operatorname{span}\{\alpha, \beta\}$. Any subset Y of Φ is contained in a unique smallest full subsystem, which we call the **full subsystem generated by Y.**

A root $\gamma \in \Phi'^+$ is called **simple in** Φ' if γ is not a positive linear combination of other roots in Φ'. Write Π' for the set of simple roots in Φ'; then Π' is a base for Φ'. We write W' for the subgroup of W generated by the reflections over the roots in Φ'. A result of Dyer [29] states that W' is a Coxeter group, with simple generators equal to the reflections over the elements of Π'. In particular, the rank of Φ' as a root system is equal to the rank of W' as a Coxeter group.

The rank of Φ' may be greater than the linear rank of Φ', even when the rank and linear rank of Φ coincide.

Example 2.3.11. Let Φ be the root system of type $\widetilde{A}_3$ and take the subsystem

$$\Phi' = \{\alpha_0 + \alpha_1 + k\delta, \alpha_1 + \alpha_2 + k\delta, \alpha_2 + \alpha_3 + k\delta, \alpha_0 + \alpha_3 + k\delta : k \in \mathbb{Z}\}.$$

The simple roots are $\{\alpha_0 + \alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_0 + \alpha_3\}$, so Φ' has rank 4, but

$$(\alpha_0 + \alpha_1) + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + (\alpha_0 + \alpha_3),$$

so $\dim \operatorname{span}(\Phi')$ is only 3. One explanation for this fact is the observation that

$$\Phi'_0 = \{\pm(\alpha_1 + \alpha_2), \pm(\alpha_2 + \alpha_3)\}$$

is a root subsystem of A_3 of type $A_1 \times A_1$. The subsystem Φ' defined above is exactly the affinization of Φ'_0, which results in Φ' having type $\widetilde{A}_1 \times \widetilde{A}_1$ and hence having rank 4.

2.3.4 Imaginary roots

Assume now that $(V, \Pi, V^*, \Pi^\vee)$ be a crystallographic root datum. We have defined the real root system to be $\Phi = W\Pi$. In applications to algebra, we will also wish to include imaginary roots. When this is the case, we write $\Delta_{\mathrm{re}} = \Phi$ for the set of real roots, Δ_{im} for the set of imaginary roots, and Δ for the set of real and imaginary roots.

Definition 2.3.12 ([43]). Let $Q = \mathbb{Z}\Phi$. We say a subset $\Delta \subseteq Q$ has the **root string property** if, for every $\alpha \in \Phi$ and $\phi \in \Delta$ so that $r_\alpha \phi = \phi + u\alpha$ with $u > 0$, the set

$$\{\phi, \phi + \alpha, \ldots, \phi + u\alpha\}$$

is contained in Δ. The **Kac–Moody root system** is the minimal $\Delta \supseteq \Phi$ having the root string property. The **imaginary roots** Δ_{im} are the elements of $\Delta \setminus \Delta_{\mathrm{re}}$.

The Kac–Moody root system is W-invariant and, setting $\Delta^+ := \Delta \cap \operatorname{cone}(\Pi)$ to be the positive roots, has the property that $\Delta = \Delta^+ \sqcup -\Delta^+$. The following is a combination of [43, Proposition 2] and [35, Proposition 1.1] (using that every crystallographic simplicial root system is also the root system of a Kac–Moody algebra, possibly after embedding into a larger vector space to make the simple coroots linearly independent). By the **height** of a root (in a simplicial root system), we always mean the sum of its coefficients in the Π basis.

Proposition 2.3.13. *Let $(V, \Pi, V^*, \Pi^\vee)$ be a crystallographic simplicial root datum. Then for all $\beta \in \Delta^+$, either*

- *There exists $\alpha_i \in \Pi$ so that $t_\alpha \beta$ has smaller height than β, or*
- *There exists $\alpha_i \in \Pi$ so that $\beta - \alpha_i \in \Delta^+$.*

Furthermore, if $\beta \in \operatorname{cone}_{\mathbb{Z}}(\Pi)$ has $\langle \alpha_i^\vee, \beta \rangle \le 0$ for all $\alpha_i \in \Pi$, then β is a root if and only if β has connected support, *in the sense that if $\beta = \sum_i c_i \alpha_i$ then $\{\alpha_i \mid c_i \neq 0\} \subseteq \Pi$ is an irreducible root datum.*

2.3.5 Kac–Moody algebras

Say a root datum $(V, \Pi, V^*, \Pi^\vee)$ is **Kac–Moody** if it is crystallographic, simplicial, dualizable, and its dual is simplicial. (These are the traditional requirements, but we could drop everything but crystallographic in this section with little change [43].)

Definition 2.3.14. The **Kac–Moody algebra** of a Kac–Moody root datum $(V, \Pi, V^*, \Pi^\vee)$ is the Lie algebra $\mathfrak{g}$ generated by the vector space V^* and elements $\{e_\alpha, f_\alpha\}_{\alpha\in\Pi}$, subject to the relations

- $[h, h'] = 0$ for $h, h' \in V^*$;
- $[h, e_\alpha] = \langle h, \alpha\rangle e_\alpha$ for $h \in V^*$ and $\alpha \in \Pi$;
- $[h, f_\alpha] = -\langle h, \alpha\rangle e_\alpha$ for $h \in V^*$ and $\alpha \in \Pi$;
- $[e_\alpha, f_\alpha] = \alpha^\vee$ for $\alpha \in \Pi$;
- $[e_\alpha, f_\beta] = 0$ for $\alpha, \beta \in \Pi$ distinct;
- $\mathrm{ad}_{e_\alpha}^{1-\langle\alpha^\vee,\beta\rangle} e_\beta = \mathrm{ad}_{f_\alpha}^{1-\langle\alpha^\vee,\beta\rangle} f_\beta = 0$ for $\alpha, \beta \in \Pi$ distinct, where ad_x is the endomorphism $y \mapsto [x, y]$.

Proposition 2.3.15. *The Kac–Moody algebra $\mathfrak{g}$ associated to a Kac–Moody root datum $(V, \Pi, V^*, \Pi^\vee)$ with Kac–Moody root system Δ admits a decomposition into finite-dimensional vector spaces*

$$\mathfrak{g} = V^* \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$$

so that if $x \in \mathfrak{g}_\alpha$ and $h \in V^*$, then $[h, x] = \langle h, \alpha\rangle x$. Furthermore, if α is a real root, then $\mathfrak{g}_\alpha$ is one-dimensional.

2.4 Lattice theory

In this section we record some definitions and facts from lattice theory that will be useful in relating extended weak order and the lattice of torsion classes. We will always be interested in *complete lattices*.

Definition 2.4.1. A **complete lattice** is a poset L so that every subset $X \subseteq L$ has a least upper bound, called its **join**, and a greatest lower bound, called its **meet**. We denote the join by $\bigvee X$ or $\bigvee_{x \in X} x$ and the meet by $\bigwedge X$ or $\bigwedge_{x \in X} x$. A (complete) **lattice homomorphism** $\eta : L_1 \to L_2$ is a function satisfying $\eta(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} \eta(x_i)$ and $\eta(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} \eta(x_i)$ for all $\{x_i\}_{i \in I} \subseteq L_1$.

Note that every finite lattice is a complete lattice.

2.4.1 Lattice quotients

Any surjective lattice homomorphism $\eta : L_1 \twoheadrightarrow L_2$ realizes L_2 as a quotient lattice of L_1 by some equivalence relation $\equiv$. The equivalence relations $\equiv$ on a complete lattice L so that the quotient map $\eta : L \twoheadrightarrow L/\equiv$ is a complete lattice homomorphism are called **complete lattice congruences**. For each equivalence class $[x] \in L/\equiv$, there are unique minimal and maximal elements in $[x]$, denoted $\pi_L^{\downarrow}(x)$ and $\pi_L^{\uparrow}(x)$, respectively. We drop the subscript L when it is clear from context. The function $\pi^{\downarrow} : L \to L$ descends to a map $\pi^{\downarrow} : L/\equiv \to L$ which is a section of the quotient map $L \twoheadrightarrow L/\equiv$. There is a similar section associated to $\pi^{\uparrow}$. These sections lead to the following characterization of complete lattice congruences.

Proposition 2.4.2 ([45, Proposition 9-5.2])**.** *Let L be a complete lattice and $\equiv$ an equivalence relation on L. Then the following are equivalent:*

(a) $\equiv$ is a complete lattice congruence on L;

(b) There is a lattice L' and a complete lattice homomorphism $\eta : L \to L'$ so that $x \equiv y$ if and only if $\eta(x) = \eta(y)$;

(c) The following conditions all hold:

- *Each equivalence class of $\equiv$ is an interval in L, and*

– *If $\pi^{\downarrow}_L(x)$ denotes the unique minimal element of $[x]_{\equiv}$, then $\pi^{\downarrow} : L \to L$ is order-preserving, and*

– *If $\pi^{\uparrow}_L(x)$ denotes the unique maximal element of $[x]_{\equiv}$, then $\pi^{\uparrow} : L \to L$ is order-preserving.*

We will also say **congruence** to mean complete lattice congruence. Given any equivalence relation $\equiv_0$ on a complete lattice L, there is a unique minimal congruence refining $\equiv_0$, the congruence **generated by** $\equiv_0$. Hence given any collection of ccongruences $\{\equiv_i\}_{i\in I}$, there is a unique minimal complete lattice congruence commonly refining each $\equiv_i$, which we call their **minimal common refinement**.

If $x \lessdot y$ is a cover relation in L and $\equiv$ is a congruence, then we say that $\equiv$ **contracts** $x \lessdot y$ if $x \equiv y$. For each cover relation $x \lessdot y$, there is a unique minimal congruence contracting $x \lessdot y$.

2.4.2 Profinite lattices

We will conjecture in Conjecture 3.0.7 that extended weak order has the property that it is a *profinite lattice*. This is a class of lattice which has not received much study, though we will find that it is a very natural property in our context. See, e.g., [4, 50] for more information about profinite lattices. At the end of this subsection we will justify the definition taken here.

Definition 2.4.3. A complete lattice congruence $\equiv$ on a complete lattice L is called **cofinite** if the quotient lattice $L/\equiv$ is finite. A complete lattice L is called **profinite** if for any $x, y \in L$ so that $x \neq y$, there is a cofinite lattice congruence $\equiv$ on L so that $x \not\equiv y$.

We summarize some results from [4].

Lemma 2.4.4. *If L' is a complete sublattice of a profinite lattice L, then L' is profinite.*

Remark 2.4.5. It is not true that complete lattice quotients of profinite lattices are profinite. For example, let L be the totally-ordered complete lattice with elements (x, i) where $x \in$

$[0,1] \subset \mathbb{R}$ and $i \in \{0,1\}$, ordered so that $(x,0) \lessdot (x,1)$ and $(x,i) < (y,j)$ if $x < y$. Then L is profinite, but admits $[0,1]$ as a complete lattice quotient, which is no longer profinite.

Lemma 2.4.6. *Let $\equiv$ be a cofinite congruence on a complete lattice L. Then for each $[y]_\equiv \in L/\equiv$, the map $x \mapsto [x]_\equiv$ induces a bijection from the lower covers of $\pi^\downarrow(y)$ in L to the lower covers of $[y]_\equiv$ in $L/\equiv$.*

Lemma 2.4.7. *If L is a profinite lattice, then for any finite list of distinct elements $x_1, \ldots, x_k \in L$ there is a cofinite congruence $\equiv$ on L so that $x_i \not\equiv x_j$ for all $i \neq j$.*

The following lemma is used implicitly in [4] but the proof is not given there, so we give it here.

Lemma 2.4.8. *Let $\{\equiv_i\}_{i\in I}$ be a collection of complete lattice congruences on a complete lattice L with the property that for any finite subset $X \subseteq L$, there exists an $i \in I$ so that $x \not\equiv_i y$ for all $x, y \in X$ with $x \neq y$. Then for any cofinite congruence $\equiv$ on L, there exists an $i \in I$ so that $\equiv_i$ is a refinement of $\equiv$.*

Proof. Let $\equiv$ be a cofinite congruence on L. If $[a]_\equiv, [b]_\equiv \in L/\equiv$ are such that $[a]_\equiv < [b]_\equiv$, then $\pi^\downarrow_\equiv(b) \not\leq \pi^\uparrow_\equiv(a)$. Hence we can pick $i \in I$ so that $\equiv_i$ distinguishes $\pi^\downarrow_\equiv(b)$ from $\pi^\downarrow_\equiv(b) \wedge \pi^\uparrow_\equiv(a)$ for each pair $[a]_\equiv, [b]_\equiv \in L/\equiv$ such that $[a]_\equiv < [b]_\equiv$. Note that this implies that $[\pi^\downarrow_\equiv(b)]_{\equiv_i} \not\leq [\pi^\uparrow_\equiv(a)]_{\equiv_i}$ whenever $[a]_\equiv < [b]_\equiv$.

The $\equiv$-equivalence classes partition L, so the images of the $\equiv$-equivalence classes in $L/\equiv_i$ cover $L/\equiv_i$. We check that if $[\pi^\downarrow_\equiv(x), \pi^\uparrow_\equiv(x)]_L$ and $[\pi^\downarrow_\equiv(y), \pi^\uparrow_\equiv(y)]_L$ are distinct $\equiv$-equivalence classes, then the images in $L/\equiv_i$ are disjoint. Indeed, if there were some $z \in [[\pi^\downarrow_\equiv(x)], [\pi^\uparrow_\equiv(x)]]_{L/\equiv_i} \cap [[\pi^\downarrow_\equiv(y)], [\pi^\uparrow_\equiv(y)]]_{L/\equiv_i}$, then that would imply

$$[\pi^\downarrow_\equiv(x \vee y)] = [\pi^\downarrow_\equiv(x)] \vee [\pi^\downarrow_\equiv(y)] \leq z \leq [\pi^\uparrow_\equiv(x)] \wedge [\pi^\uparrow_\equiv(y)] = [\pi^\uparrow_\equiv(x \wedge y)].$$

Since $[x \wedge y]_\equiv \leq [x \vee y]_\equiv$, we have either $[x \wedge y]_\equiv = [x \vee y]_\equiv$ (in which case $[x]_\equiv = [y]_\equiv$ which is impossible) or else $[x \wedge y]_\equiv < [x \vee y]_\equiv$. In the latter case, we have chosen $\equiv_i$ so that $[\pi^\downarrow_\equiv(x \vee y)]_{\equiv_i} \not\leq [\pi^\uparrow_\equiv(x \wedge y)]_{\equiv_i}$, so this is also impossible. Hence distinct $\equiv$-equivalence classes are mapped to disjoint sets in $L/{\equiv_i}$. Furthermore, the maps $\pi^{\uparrow}_{\equiv}, \pi^{\downarrow}_{\equiv} : L/{\equiv_i} \to L/{\equiv_i}$ are order-preserving, since they lift to order-preserving maps $L \to L$. Hence by Proposition 2.4.2, there is a lattice congruence on L_i with classes given by the parts $[[\pi^{\downarrow}_{\equiv}(x)], [\pi^{\uparrow}_{\equiv}(x)]]_{L/\equiv_i}$ giving a factorization $L \twoheadrightarrow L/{\equiv_i} \twoheadrightarrow L/{\equiv}$. It follows that $\equiv_i$ refines $\equiv$ as desired. □

Profiniteness is a property

Here we justify the definition we take for profinite lattices. This subsubsection may be skipped by the reader uninterested in the categorical notion of profiniteness.

There are many examples of profinite objects in mathematics. Common examples include profinite sets, profinite groups, and complete local rings over a finite field with the adic topology. In general, a profinite object should be a formal inverse limit of a cofiltered system of finite objects (this is also called a *pro-object* in the category of finite objects). Let $\mathcal{C}$ be the category of finite objects (in our case, finite lattices). In particular, we assume $\mathcal{C}$ is small. We shall freely identify objects $X \in \mathcal{C}$ with the representable functor $\mathrm{Hom}(X, -)$, viewed as an object of the category $\widehat{\mathcal{C}} := \mathbf{Funct}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$. By the Yoneda lemma, for any functor $F : \mathcal{C} \to \mathbf{Set}$ and object $X \in \mathcal{C}$, there is a bijection between $F(X)$ and $\mathrm{Hom}_{\widehat{\mathcal{C}}}(F, X)$. Hence for any $x \in F(X)$ we may write $F \xrightarrow{x} X$ for the corresponding morphism from F to X. A **pro-object** is a functor $F : \mathcal{C} \to \mathbf{Set}$ so that:

(a) For any $X, Y \in \mathcal{C}$ and $x \in F(X), y \in F(Y)$, there is some object $Z \in \mathcal{C}$, element $z \in F(Z)$, and morphisms $\alpha : Z \to X$, $\beta : Z \to Y$, so that the following diagram commutes.

$$\begin{tikzcd} & & X \\ F \arrow[urr, "x"] \arrow[r, "z"] \arrow[drr, "y"'] & Z \arrow[ur, "\alpha"'] \arrow[dr, "\beta"] & \\ & & Y \end{tikzcd}$$

(b) For any parallel morphisms $\alpha, \beta : X \rightrightarrows Y$ in $\mathcal{C}$ and $x \in F(X)$, if $F \xrightarrow{x} X \xrightarrow{\alpha} Y$ and $F \xrightarrow{x} X \xrightarrow{\beta} Y$ compose to the same morphism, then there is an object $Z \in \mathcal{C}$, element

$z \in F(Z)$, and morphism $\gamma : Z \to X$ so that $\gamma : Z \to X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$ compose to the same morphism and so that $F \xrightarrow{z} Z \xrightarrow{\gamma} X$ composes to x.

Write $\mathbf{Pro}(\mathcal{C})$ for the category of pro-objects in $\mathcal{C}$ (a full subcategory of $\widehat{\mathcal{C}}$). Often we can realize pro-objects more explicitly. We say an object X in a category $\mathcal{D}$ is **cocompact** if $\mathrm{Hom}_{\mathcal{D}}(-, X)$ preserves cofiltered limits. Whenever $\mathcal{C}$ is a full subcategory of another category $\mathcal{D}$ which admits cofiltered limits and so that $\mathcal{C}$ consists of *cocompact* objects of $\mathcal{D}$, then there is a canonical embedding of $\mathbf{Pro}(\mathcal{C})$ as a full subcategory of $\mathcal{D}$. The essential image is the subcategory of $\mathcal{D}$ consisting of objects which are cofiltered limits of objects of $\mathcal{C}$; equivalently, objects which are limits of objects of $\mathcal{C}$ and satisfy properties (a) and (b), where we interpret $F(X)$ to mean $\mathrm{Hom}_{\mathcal{D}}(F, X)$.

Example 2.4.9. Consider the category $\mathcal{C} = \mathbf{FinSet}$ of finite sets. The category $\mathcal{D} = \mathbf{CHaus}$ of compact Hausdorff spaces contains $\mathcal{C}$ as a full subcategory (the subcategory of finite discrete topological spaces). Furthermore, each object of $\mathcal{C}$ is cocompact in $\mathcal{D}$. Hence, $\mathbf{Pro}(\mathbf{Set})$ is a full subcategory of $\mathcal{D}$. Its essential image is the category of **Stone spaces**: compact Hausdorff totally disconnected spaces. Hence the category of profinite sets $\mathbf{Pro}(\mathbf{Set})$ is equivalent to the category of Stone spaces. Often profinite sets are *defined* to be Stone spaces to circumvent the category theory; this is analogous to the circumvention taken in Definition 2.4.3.

We now specialize to the case $\mathcal{C} = \mathbf{FinLat}$ and $\mathcal{D} = \mathbf{ProfLat}$, the categories of finite lattices and profinite lattices (as defined in Definition 2.4.3), respectively. Morphisms in both categories are complete lattice homomorphisms.

Lemma 2.4.10. ProfLat *admits small limits.*

Proof. We will show that the limit of profinite lattices in the category of all complete lattices is in fact a profinite lattice. It is enough to check this for products and equalizers. The equalizer of any two morphisms in **ProfLat** is in particular a complete sublattice of a profinite lattice, so by Lemma 2.4.4 it is profinite. Hence it remains to check that a product of profinite lattices is profinite. Indeed, if $x, y \in \prod_{i \in I} L_i$ are distinct, then there is some i so that $x_i \neq y_i$. Since L_i is profinite, there is a finite quotient $L_i \twoheadrightarrow F$ so that the images of x_i and y_i are distinct. Composing with the projection $\prod_{i \in I} L_i \twoheadrightarrow L_i$ gives the desired finite quotient. $\square$

Proposition 2.4.11. *The finite lattices are cocompact objects of $\mathcal{D} = \mathbf{ProfLat}$. The resulting embedding of the category $\mathbf{Pro}(\mathbf{FinLat})$ into $\mathcal{D}$ is an equivalence of categories $\mathbf{Pro}(\mathbf{FinLat}) \xrightarrow{\sim} \mathbf{ProfLat}$.*

Proof. First we check that every profinite lattice is a cofiltered limit of finite lattices. Let L be a profinite lattice, and consider the category I of arrows $L \to X_i$, where $X_i \in \mathbf{FinLat}$. We claim this category is cofiltered; i.e. that (a) and (b) are satisfied. Indeed, given $L \to X_i$ and $L \to X_j$, then $X_i \times X_j$ is a finite lattice and the induced map $L \to X_i \times X_j$ fulfils the requirement of (a). Given $L \to X_i \rightrightarrows X_j$, let X be the equalizer of $X_i \rightrightarrows X_j$. Then X is a finite lattice and the induced map $L \to X$ fulfils the requirement of (b). So the arrow category is cofiltered; we now view it as a diagram in $\mathbf{FinLat}$. Let $L' = \varprojlim_I X_i$. We must check that the induced map $L \to L'$ is an isomorphism. Indeed, $L \to L'$ is injective since by Definition 2.4.3 we may distinguish any two elements of L via a finite quotient. To see surjectivity, let $x \in L'$. Then for each arrow $L \to X_i$, there is an associated $x_i \in X_i$. Each surjective arrow can be viewed as a lattice quotient $L \twoheadrightarrow X_i$, so we may take $\pi_L^{\downarrow}(x_i)$. Then the join $\bigvee_{i \in I} \pi_L^{\downarrow}(x_i)$ maps to $x \in L'$, so $L \to L'$ is an isomorphism.

Now we check that any cofiltered limit of finite lattices in $\mathbf{Pro}(\mathcal{C})$ is also the cofiltered limit of finite lattices with all transition maps given by surjections. By [48, Tag 0594], it is enough to verify that any cofiltered diagram $I \to \mathcal{C}$ satisfies the *Mittag–Leffler condition*: for any X_i, there is a $X_j \xrightarrow{\alpha_j} X_i$ in the diagram so that for any $X_k \xrightarrow{\alpha_k} X_j$ in the diagram, the image of α_j equals the image of $\alpha_j \circ \alpha_k$. This follows since, for each i, there must be a j attaining the minimal size of the image of $X_j \to X_i$ among all maps in the diagram. Then any $X_k \to X_j$ must map to that same image in X_i.

Hence to check that X is cocompact in $\mathcal{D}$ it is enough to check that $\mathrm{Hom}_{\mathcal{D}}(-, X)$ commutes with cofiltered limits in $\mathcal{D}$ of objects in $\mathcal{C}$, where the transition maps are all surjections.

Let $I \to \mathcal{C}$ be a cofiltered diagram with all transition maps surjective. Let $L := \varprojlim_{i \in I} L_i$ be its colimit in $\mathcal{D}$. Then we wish to show for a finite lattice F that the canonical map

$$\Psi : \varinjlim_{i \in I} \mathrm{Hom}_{\mathcal{D}}(L_i, F) \to \mathrm{Hom}_{\mathcal{D}}(L, F)$$

is an isomorphism. First note that each induced map $L \to L_i$ is surjective, since for $x \in L_i$ we get an element $\pi^{\downarrow}_{L_j}(x)$ for each map $L_j \to L_i$ in the diagram. This gives a compatible system of elements of lattices in the diagram, and hence an element of the limit L. Now, let $\phi_1 : L_{i_1} \to F, \phi_2 : L_{i_2} \to F$ be elements of the left hand side so that $\Psi(\phi_1) = \Psi(\phi_2)$. Then there is some common refinement L_i of L_{i_1}, L_{i_2}. Since $L \to L_i$ is surjective, the restrictions of ϕ_1 and ϕ_2 to L_i are equal, so ϕ_1 and ϕ_2 are equal in the colimit.

Finally, let $\phi : L \to F$ be an element of the right hand side. Then the image of ϕ is a finite lattice quotient F' of L. Let $\equiv$ be the associated cofinite congruence on L. Each map $L \to L_i$ also induces a cofinite congruence $\equiv_i$ on L, and the family $\{\equiv_i\}_{i \in I}$ satisfies the hypothesis of Lemma 2.4.8. Hence there is some $i \in I$ so that $\equiv_i$ refines $\equiv$. Then $\equiv$ descends to an equivalence relation on L_i so that $L_i/\equiv$ is isomorphic to F'. Taking $\phi_i : L_i \twoheadrightarrow F' \hookrightarrow F$, we find that $\Psi(\phi_i) = \phi$, so we are done.

□

Remark 2.4.12. One consequence of this proposition is the fact that each complete lattice can have the structure of a profinite lattice in at most one way. This was also shown in [50] using topological lattices. Compare with the case of profinite sets, where there are many inequivalent profinite structures on any infinite set. For example, the Stone spaces $\mathbb{N} \cup \{\infty\}$ and $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ with their order topologies are non-isomorphic profinite sets, even though there is a bijection between their underlying sets.

2.4.3 Compact elements

In this subsection we will discuss three notions of finiteness for an element of a complete lattice. For a general complete lattice these notions are distinct, but for profinite lattices they are closely related. The results in this subsection are proven in [4], and the terms *widely generated* and *finitely generated* are introduced there.

Definition 2.4.13. Let L be a complete lattice and let $x \in L$. Then we make the following definitions:

- x is **compact** if, for any subset $X \subseteq L$ so that $x \leq \bigvee X$, there is some finite subset $X' \subseteq X$ so that $x \leq \bigvee X'$;
- x is **widely generated** if for all $y < x$ there exists a cover relation $x' \lessdot x$ so that $y \leq x'$;
- x is **finitely generated** if there is a cofinite congruence $\equiv$ on L so that $x = \pi^{\downarrow}(x)$;
- x is a **complete join-irreducible** (**cJI**) if for any $X \subseteq L$ so that $\bigvee X = x$, then $x \in X$.

If every element of L is the join of a set of compact elements, then L is an **algebraic lattice**.

Any cJI j has at most one lower cover, since for any element of a lattice j, if $j_1, j_2 \lessdot j$ are two distinct lower covers of j, then $j_1 \vee j_2$. Furthermore, j has at least one lower cover, since the element $j_* := \bigvee_{x<j} x$ is a lower cover of j. In a finite lattice, having exactly one lower cover is also sufficient for being a complete join-irreducible. In infinite lattices this is no longer true.

Example 2.4.14. Consider the subposets $L_1 = \{0, 1\}$ and $L_2 = [0, 1]$ of the real numbers $\mathbb{R}$. Then in the product lattice $L_1 \times L_2$, the element $(1, 1)$ covers only the element $(0, 1)$, but it is not a complete join-irreducible.

The correct characterization of complete join-irreducibles in an infinite lattice is the following.

Lemma 2.4.15. *An element j of a complete lattice L is a cJI if and only if j is widely generated and there is a unique lower cover $j_* \lessdot j$.*

We show the following in [4].

Proposition 2.4.16. *The following are equivalent, for an element x of a profinite lattice L:*

(a) x is compact;

(b) x is finitely generated;

(c) x is widely generated and x covers finitely many elements; and

(d) x is the join of finitely many complete join-irreducibles.

Proposition 2.4.17. *If L is a profinite lattice, then every element of L is the join of a set of complete join-irreducibles. In particular, L is an algebraic lattice.*

Remark 2.4.18. It is not true for an arbitrary lattice that every element is the join of a set of cJIs. For instance, consider the closed interval $L = [0, 1]$ viewed as a subposet of the real numbers $\mathbb{R}$. Then there are *no* cJIs in L, so the only element which is a join of cJIs is 0 (the join of the empty set).

2.4.4 Semidistributive lattices

Definition 2.4.19. A complete lattice L is **completely join semidistributive** if, for every $y, z \in L$, whenever $X \subseteq L$ satisfies $x \vee y = z$ for all $x \in X$, then also $(\bigwedge X) \vee y = z$. It is **completely meet semidistributive** if, for every $y, z \in L$, whenever $X \subseteq L$ satisfies $x \wedge y = z$ for all $x \in X$, then also $(\bigvee X) \wedge y = z$. If L is completely meet and join semidistributive, then we say L is **completely semidistributive**. If these conditions hold only when X is a finite set, then we say L is **semidistributive**.

For finite lattices, it is known that semidistributivity is equivalent to the existence of *canonical meet* and join representations.

Definition 2.4.20. Let L be a complete lattice and let $U, V \subseteq L$. We write $U \ll V$ to mean that for every $u \in U$, there is some $v \in V$ so that $u \leq v$. We say that $\bigvee U = x$ is an

irredundant join representation of x if there is no proper subset $U' \subsetneq U$ so that $\bigvee U' = x$. We say that $\bigvee U = x$ is the **canonical join representation** of x if it is irredundant and for any $V \subseteq L$ so that $\bigvee V = x$, we have $U \ll V$.

For profinite lattices, semidistributivity and existence of canonical meet and join representations are closely related, although it is no longer true that every element of a semidistributive profinite lattice has such a representation.

Proposition 2.4.21. *Let L be a profinite lattice. Then the following are equivalent:*

(a) L is join semidistributive;

(b) L is completely join semidistributive;

(c) For every cofinite congruence $\equiv$ on L, the quotient lattice $L/\equiv$ is join semidistributive;

(d) Every widely generated element of L has a canonical join representation;

(e) Every finitely generated element of L has a canonical join representation;

(f) For every pair of distinct elements x, y of L, there is a cofinite congruence $\equiv$ with $x \not\equiv y$ so that $L/\equiv$ is join semidistributive.

(g) For every finite subset X of L, there is a cofinite congruence $\equiv$ with $x \not\equiv y$ for distinct $x, y \in X$ and so that $L/\equiv$ is join semidistributive.

Proof. The equivalence of (a-e) appears in [4], so we prove their equivalence with (f) and (g). Evidently (c) implies (g) implies (f). Also, (f) implies (g): if $X = \{x_1, \dots, x_k\}$ then let $\equiv_{ij}$ be a cofinite congruence distinguishing x_i and x_j and so that $L/\equiv_{ij}$ is join semidistributive. The quotient maps $L \to L/\equiv_{ij}$ induce a map $L \to \prod_{i,j} L/\equiv_{ij}$. Join semidistributive lattices are closed under products and sublattices, so the image of L is a finite semidistributive lattice. The quotient map from L to its image defines a cofinite congruence satisfying the requirement for (g). Finally, (g) implies (c) by Lemma 2.4.8 and the fact that join semidistributivity of finite lattices is preserved under quotients [45, Proposition 9-5.26]. $\square$

2.4.5 Fundamental theorem of profinite semidistributive lattices

We recall here some notions and conventions from [46], regarding the "fundamental theorem of finite semidistributive lattices".

Definition 2.4.22. A **two-acyclic factorization system** consists of a tuple $(Ш, \to, \twoheadrightarrow, \hookrightarrow)$, where:

- $Ш$ (pronounced "sha") is a set, and
- $\to, \twoheadrightarrow, \hookrightarrow$ are binary relations on $Ш$ which satisfy:

(Factorization I) There is an arrow $x \to z$ if and only if there are arrows $x \twoheadrightarrow y \hookrightarrow z$ for some $y \in Ш$;

(Factorization IIa) There is an arrow $x \twoheadrightarrow y$ if and only if for all arrows $y \to z$ there is also an arrow $x \to z$;

(Factorization IIb) There is an arrow $x \hookrightarrow y$ if and only if for all arrows $z \to x$ there is also an arrow $x \to y$;

(Order condition) The relations $\hookrightarrow$ and $\twoheadrightarrow$ are partial orders; and

(Brick condition) If $x \twoheadrightarrow y \hookrightarrow x$ then $x = y$.

Note that the factorization condition means that $(Ш, \to, \twoheadrightarrow, \hookrightarrow)$ is determined by the data of either $(Ш, \to)$ or $(Ш, \twoheadrightarrow, \hookrightarrow)$.

Example 2.4.23. Let $\mathcal{C}$ be a finite-length abelian category (such as the category of finite-dimensional modules over an algebra). Let $Ш$ be the set of isomorphism classes of **bricks** of $\mathcal{C}$: the objects B so that $\mathrm{End}_{\mathcal{C}}(B)$ is a division algebra (meaning that every non-zero endomorphism of B is invertible). Then there is a unique two-acyclic factorization system $(Ш, \to, \twoheadrightarrow, \hookrightarrow)$ so that $A \to B$ if and only if $\mathrm{Hom}_{\mathcal{C}}(A, B) \neq 0$. There is an arrow $A \hookrightarrow B$ if and only if A has a filtration $0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_k = A$ so that each subquotient A_{i+1}/A_i is (isomorphic to) a submodule of B. Dually, there is an arrow $A \twoheadrightarrow B$ if and only if B has a filtration $0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_k = B$ so that each subquotient B_{k+1}/B_k is (isomorphic to) a quotient of A.

Given a relation $\to$ on a set Ш and subsets $X, Y \subseteq$ Ш, we define $X^\perp = \{y \in \text{Ш} \mid \forall x \in X,\ x \not\to y\}$ and ${}^\perp Y = \{x \in \text{Ш} \mid \forall y \in Y, x \not\to y\}$.

Given a two-acyclic factorization system $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$, there is an associated lattice $\mathrm{Pairs}(\to)$, the **lattice of maximal orthogonal pairs**. Its elements are pairs (X, Y) where $X^\perp = Y$ and $X = {}^\perp Y$, ordered so that $(X_1, Y_1) \leq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ (equivalently, $Y_1 \supseteq Y_2$). Furthermore, the original two-acyclic factorization system is recoverable up to equivalence from the lattice $\mathrm{Pairs}(\to)$. The following is a main result of [46].

Proposition 2.4.24. *Up to isomorphism, the finite semidistributive lattices are exactly the lattices of maximal orthogonal pairs of a two-acyclic factorization system* $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ *so that* $|\text{Ш}|$ *is finite.*

We show the following.

Theorem 2.4.25. *Let L be a profinite semidistributive lattice. Then there is a two-acyclic factorization system* $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ *so that L is isomorphic to* $\mathrm{Pairs}(\to)$.

Proof. In the language of [46], it is equivalent to show that L is a *well-separated κ-lattice.* Since profiniteness is preserved under duality, Proposition 2.4.16 implies both L and L^{op} are algebraic. Since L is completely semidistributive, by [46, Theorem 3.1] L is a κ-lattice. By that same theorem, to check L is well-separated, it is enough to show that whenever $x < y$ and $x \wedge z < y \wedge z$, then there is a cover relation $x \leq u \lessdot v \leq y$ with $u \wedge z < v \wedge z$ (and the dual statement). To see this, consider any cofinite congruence so that $x \wedge z \not\equiv y \wedge z$. Let $\mathcal{C}$ be a saturated chain in L from x to y. Set $u = \bigvee\{u \in \mathcal{C} \mid u \equiv x \wedge z\}$. Then u is covered by a unique element v in $\mathcal{C}$. (The restriction of $\equiv$ to $\mathcal{C}$ is a complete lattice congruence, and $u = \pi^{\uparrow}_{\mathcal{C}}(u)$, so v is the minimal element of the $\equiv$-equivalence class covering that of u.) Then $x \leq u \lessdot v \leq y$ and $[u \wedge z]_\equiv < [v \wedge z]_\equiv$. The dual statement follows from a dual argument, so we conclude that L is a well-separated κ-lattice. $\square$

Let $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ be a two-acyclic factorization system. Then certain quotients of $\mathrm{Pairs}(\to)$ can be described in terms of the factorization system. We say a complete lattice L is **weakly atomic** if for any $x < y$, there exists a cover relation $u \lessdot v$ so that $x \leq u \lessdot v \leq y$. By [46, Theorem 3.1], $\mathrm{Pairs}(\to)$ is always weakly atomic. Given an arrow $x \to z$, an **image** of $x \to z$ is a $y \in \text{Ш}$ so that $x \twoheadrightarrow y \hookrightarrow z$ and so that if $y' \in \text{Ш}$ with $y \twoheadrightarrow y' \hookrightarrow z$ then $y = y'$. Similarly, a **co-image** of $x \to z$ is a $y \in \text{Ш}$ so that $x \twoheadrightarrow y \hookrightarrow z$ and so that if $y' \in \text{Ш}$ with $x \twoheadrightarrow y' \hookrightarrow y$ then $y = y'$.

Lemma 2.4.26. *If* $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ *is a two-acyclic factorization system so that* $\mathrm{Pairs}(\to)$ *is profinite, then any arrow* $x \to z$ *has an image and a coimage.*

Proof. We check that $x \to y$ has an image; the case of a co-image is dual. This means that if x, z are cJIs and $x \not\leq \kappa(z)$, then there is some minimal cJI $y \leq x$ with the property that $\kappa(y) \geq \kappa(z)$. Let $\equiv$ be a cofinite congruence so that $x \not\equiv x_*, \kappa(z) \not\equiv \kappa(z)^*$, and $x \wedge \kappa(z) \not\equiv x$. Then $[\kappa(x)]_\equiv = \kappa([x]_\equiv)$. Let $[y]$ be an image of $[x]_\equiv \to [z]_\equiv$ in $L/\equiv$. Then $\pi^\downarrow(y)$ is an image of $x \to z$. $\square$

Let $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ be a two-acyclic factorization system. Then for $x, y \in \text{Ш}$, we say that x **directly forces** y if either

- $x \hookrightarrow y$ and if $x \twoheadrightarrow x' \hookrightarrow y$, then $x = x'$; or
- $y \twoheadrightarrow x$ and if $y \twoheadrightarrow x' \hookrightarrow x$, then $x = x'$.

Let $\text{Ш}'$ be a subset of Ш. We say $\text{Ш}'$ is **forcing-closed** if whenever $x \in \text{Ш}'$ and x directly forces $y \in \text{Ш}$, then $y \in \text{Ш}'$. We say $\text{Ш}'$ is **forcing-open** if $\text{Ш} \setminus \text{Ш}'$ is forcing-closed. The following is [46, Theorem 6.3].

Proposition 2.4.27. *Let* $(\text{Ш}, \to, \twoheadrightarrow, \hookrightarrow)$ *be a two-acyclic factorization system so that every arrow* $x \to y$ *has an image and a co-image. For any forcing-open subset* $\text{Ш}' \subseteq \text{Ш}$*, the tuple* $(\text{Ш}', \to, \twoheadrightarrow, \hookrightarrow)$ *is a two-acyclic factorization system and the restriction map* $\mathrm{Pairs}_{\text{Ш}}(\to) \to \mathrm{Pairs}_{\text{Ш}'}(\to)$ *is a complete lattice quotient. This gives a bijection between forcing-open subsets of* Ш *and weakly atomic complete lattice quotients of* $\mathrm{Pairs}(\to)$.

We refer to taking the lattice quotient $\mathrm{Pairs}_{Ш}(\to) \twoheadrightarrow \mathrm{Pairs}_{Ш'}(\to)$ as **contracting** the elements of $Ш \setminus Ш'$. The elements of $Ш'$ are the **uncontracted** elements. Combining Theorem 2.4.25, Lemma 2.4.26, and Proposition 2.4.27, we have the following.

Corollary 2.4.28. *Let* $(Ш, \to, \twoheadrightarrow, \hookrightarrow)$ *be a two-acyclic factorization system. Then* $\mathrm{Pairs}(\to)$ *is profinite if and only if every arrow has an image and co-image and every* $x \in Ш$ *is contained in a finite forcing-open subset* $Ш' \subseteq Ш$.

2.4.6 Polygonal lattices and lattice quotients

In this subsection we recall the notion of a *polygonal lattice*, which is a property of the weak order on a finite Coxeter group and is important for studying the lattice quotients of weak order (see, e.g., [45, Section 9-6]).

Definition 2.4.29. A bounded poset P is a **polygon** if P has exactly two maximal chains. Let L be a finite lattice. Then L is **polygonal** if for every x, y in L so that either x and y cover the same element or x and y are covered by the same element, the interval $[x \wedge y, x \vee y]$ is a polygon.

We say that a profinite lattice L is **pro-polygonal** if for every cofinite congruence $\equiv$ on L, the finite lattice $L/{\equiv}$ is polygonal.

Lemma 2.4.30. *A profinite lattice* L *is pro-polygonal if and only if for any finite list of distinct elements* $x_1, \ldots, x_k \in L$*, there is a cofinite congruence* $\equiv$ *so that* $x_i \not\equiv x_j$ *when* $i \neq j$ *and* $L/{\equiv}$ *is polygonal.*

Proof. The forwards direction is clear from Lemma 2.4.7. The other direction follows from Lemma 2.4.8 and the fact that polygonality is preserved by quotients [45, Proposition 9-6.9]. $\square$

Chapter 3

Biclosed sets of roots

Let Φ be a root system associated to a Coxeter group W, with root datum $(V, \Pi, V^*, \Pi^\vee)$ and positive roots Φ^+. Matthew Dyer, in the course of studying Kazhdan–Lusztig theory, introduced a family of subsets of Φ^+, the *biclosed sets*, which behave like the intersections of systems of positive roots with Φ^+. Biclosed sets generalize similar families (called *biconvex sets* or *compatibly ordered sets*) which had been studied in the earlier works of Ito [34] and Cellini–Papi [21]. They also include the **infinite reduced words** studied by Lam and Pylyavsky [40]. Biclosed sets are the subject of many open conjectures [27], some of which will be discussed below.

Definition 3.0.1. Let $B \subseteq \Phi^+$ be a set of positive roots. We say that:

- B is **closed** in Φ^+ if, whenever α and $\beta \in B$, and $\gamma \in \operatorname{cone}(\alpha, \beta) \cap \Phi^+$, then $\gamma \in B$. We say that B is **coclosed** in Φ^+ if $\Phi^+ \setminus B$ is closed in Φ^+.
- B is **convex** in Φ^+ if $\operatorname{cone}(B) \cap \Phi^+ = B$. We say that B is **coconvex** in Φ^+ if $\Phi^+ \setminus B$ is convex in Φ^+.
- B is **biclosed** in Φ^+ if B is closed and coclosed in Φ^+.
- B is **biconvex** in Φ^+ if B is convex and coconvex in Φ^+.
- B is **weakly separable** in Φ^+ if $\operatorname{cone}(B) \cap \operatorname{cone}(\Phi^+ \setminus B) = \{0\}$.

- B is **strictly separable** in Φ^+ if there is a dual vector $\theta \in V^*$ such that $B = \{\alpha \in \Phi^+ : \langle \theta, \alpha \rangle < 0\}$ and $\Phi^+ \setminus B = \{\alpha \in \Phi^+ : \langle \theta, \alpha \rangle > 0\}$.

We have the implications

$$\text{strictly separable} \Rightarrow \text{weakly separable} \Rightarrow \text{biconvex} \Rightarrow \text{biclosed}.$$

We write $\mathrm{Bic}(\Phi^+)$ for the collection of biclosed sets in Φ^+, viewed as a poset under containment order. This poset is called the **extended weak order**. It turns out to be an invariant of the Coxeter group W, in the following sense. The map $t \mapsto \alpha_t$ is a bijection $T \to \Phi^+$. Hence each biclosed set $B \subseteq \Phi^+$ determines a set of reflections $\{t \in T \mid \alpha_t \in B\} \subseteq T$. Then the collection of subsets of T arising in this way is independent of the choice of root data $(V, \Pi, V^*, \Pi^\vee)$ defining a root system for W. When we are ambivalent about the precise choice of root system, we will sometimes write $\mathrm{Bic}(W)$ to mean the extended weak order, realized as the poset of biclosed sets in some root system for W.

Remark 3.0.2. Unlike biclosed sets, the biconvex sets, weakly separable sets, and strictly separable sets *are* all sensitive to the choice of root data $(V, \Pi, V^*, \Pi^\vee)$ defining a root system for W.

Recall that for $w \in W$, the *inversion set* of w is $N(w) = \{\alpha \in \Phi^+ \mid w^{-1}\alpha \in -\Phi^+\}$. Dyer showed the following, characterizing the finite biclosed sets. Cellini–Papi [21] and Ito [34] showed earlier a similar result when W is an affine Weyl group.

Proposition 3.0.3. *Let $B \subseteq \Phi^+$ be a finite set of a positive roots. Then B is biclosed if and only if B is separable if and only if $B = N(w)$ for some $w \in W$.*

Dyer has two main conjectures on biclosed sets [28]. The first asserts that biclosed sets are the same as "initial sections of reflection orders", which is a family appearing in applications of biclosed sets to Kazhdan–Lusztig theory [25, 31, 32] (see also Chapter 5).

Definition 3.0.4. Let W be a Coxeter group with reflections T. Given a total ordering $\prec$ on T, an **initial section** is a subset $A \subseteq \Phi^+$ so that if $\alpha_t \in A$ and $t' \prec t$, then $\alpha_{t'} \in A$. A

reflection order is a total ordering $\prec$ on T so that every initial section of $\prec$ is biclosed in Φ^+.

Evidently, every initial section of a reflection order is a biclosed set. Dyer's first conjecture implies that every biclosed set is the initial section of some reflection order (first conjectured in [31, Remark 2.12]).

Conjecture 3.0.5 (Dyer's Conjecture A)**.** *Let $\mathcal{C}$ be a maximal chain in* $\mathrm{Bic}(\Phi^+)$*. Then for any $\alpha, \beta \in \Phi^+$, there exists a biclosed set $B \in \mathcal{C}$ so that either $\alpha \in B$ and $\beta \notin B$, or $\alpha \notin B$ and $\beta \in B$.*

Assuming the conjecture, then given a maximal chain $\mathcal{C}$ of $\mathrm{Bic}(\Phi^+)$, there is a unique total ordering $\prec$ of T so that $t' \prec t$ if and only if there exists a biclosed set $B \in \mathcal{C}$ with $\alpha_{t'} \in B$ and $\alpha_t \notin B$. Equivalently, $\prec$ is the unique reflection order whose collection of initial sections is $\mathcal{C}$. Conjecture A also implies that (and is in fact equivalent to) whenever $B_1 \lessdot B_2$ is a cover relation in $\mathrm{Bic}(\Phi^+)$, then $|B_2 \setminus B_1| = 1$.

Dyer's second conjecture is about the poset structure of $\mathrm{Bic}(W)$. It generalizes the well-known fact that containment order on inversion sets (the *weak order*) of a finite Coxeter group is a lattice.

Conjecture 3.0.6 (Dyer's Conjecture B)**.** *For any Coxeter group W, the extended weak order* $\mathrm{Bic}(W)$ *is a complete lattice.*

In [7], the current author and David Speyer reduced Conjectures 3.0.5 and 3.0.6 to showing the existence of *suitable orders* on Φ^+. We will discuss this approach later in this chapter, where we will also extend it to apply to the following conjecture, raised in [4].

Conjecture 3.0.7. *For any Coxeter group W, the extended weak order* $\mathrm{Bic}(W)$ *is a profinite semidistributive lattice.*

3.1 Quasiregions

In this section, we will reinterpret the notion of a biclosed set in terms of the hyperplane arrangement $\mathcal{A} = \{\alpha^\perp \subseteq V^* \mid \alpha \in \Phi^+\}$ dual to Φ^+. Then $\mathcal{A} = \mathcal{A}_{(W,S)}$ is the reflection arrangement associated to the action of W on V^*. We call an arrangement arising this way (for some choice of separable root system associated to W) a **Coxeter arrangement** of W. A Coxeter arrangement is a based reflection arrangement in a canonical way: take a functional $\rho \in V^*$ so that $\langle \rho, \alpha_i \rangle > 0$ for all $i \in I$, and let the base region of $\mathcal{A}$ be the (strict) region R_0 having ρ as a point.

Recall the definition of pre-region and region for a hyperplane arrangement from Section 2.1. The regions of $\mathcal{A}$ correspond to weakly separable sets, by sending a region R to the set $\{\alpha \in \Phi^+ \mid \alpha^\perp \in \mathcal{S}(R)\} \subseteq \Phi^+$. The pre-regions which correspond to biclosed sets are called *quasiregions*.

Definition 3.1.1. Let R be a pre-region of a hyperplane arrangement $\mathcal{A}$. Then R is a **quasiregion** if for every $H_1, H_2, H_3 \in \mathcal{A}$, the intersection $H_1^R \cap H_2^R \cap H_3^R$ is non-empty.

Lemma 3.1.2. *If $\mathcal{A}$ is a hyperplane arrangement, then a pre-region R of $\mathcal{A}$ is a quasiregion if and only if R restricts to a region in every rank 2 subarrangement of $\mathcal{A}$.*

Proof. Let R be a pre-region of $\mathcal{A}$ and let $\mathcal{A}'$ be a rank 2 subarrangement of $\mathcal{A}$. If the restriction of R to $\mathcal{A}'$ is not a region, then there is some finite set $\{H_1, \ldots, H_k\} \subseteq \mathcal{A}'$ so that $H_1^R \cap \cdots \cap H_k^R = \varnothing$. Take k to be minimal. Then R restricts to a region R' of the rank 2 arrangement $\{H_1, \ldots, H_{k-1}\}$. The region R' has two walls H_{i_1}, H_{i_2}, and we have $H_{i_1}^R \cap H_{i_2}^R \cap H_k^R = \varnothing$, so R is not a quasiregion.

Conversely, if R is a pre-region which restricts to a region of every rank 2 subarrangement, then for any triple of hyperplanes H_1, H_2, H_3, either $\{H_1, H_2, H_3\}$ is a rank 2 arrangement and so $H_1^R \cap H_2^R \cap H_3^R \neq \varnothing$ by assumption, or else it is a rank 3 arrangement and every possible intersection of half-spaces is non-empty. $\square$

We will be particularly interested in hyperplane arrangements which satisfy the dual property to “every biclosed set is weakly separable”.

Definition 3.1.3. A hyperplane arrangement $\mathcal{A}$ is a **clean arrangement** if every quasiregion of $\mathcal{A}$ is a region.

The term *clean* was introduced in [7], but the notion has been studied independently several times. In particular, finite arrangements that are simplicial or supersolvable were shown to be clean in [41, Chapter 4]. More generally, finite $K(\pi, 1)$ arrangements were recently shown to be clean arrangements in [51].

Remark 3.1.4. Whether an arrangement $\mathcal{A}$ is clean or not is independent of the choice of base region. This is not as clear in the dual language, where cleanliness is the assertion that every biclosed set is weakly separable, and changing the base region corresponds to replacing Φ^+ with a different set of positive roots for Φ.

The following is a weakening of [7, Conjecture 1.3], which will be proven in many cases in a forthcoming joint work.

Conjecture 3.1.5. *If W is a rank 3 Coxeter group, then any Coxeter arrangement of W is clean.*

We have proven this in finite and affine type in [7].

Proposition 3.1.6. *If W is a finite Coxeter group or a rank 3 affine Coxeter group, then any Coxeter arrangement of W is clean.*

Given a base region, the quasiregions of $\mathcal{A}$ can be ordered in a way corresponding to the containment order on biclosed sets. The **poset of quasiregions** is the poset $\mathsf{Quasi}(\mathcal{A})$ which orders the quasiregions so that $R_1 \leq R_2$ if $\mathcal{S}(R_1) \subseteq \mathcal{S}(R_2)$. Evidently, we have the following.

Lemma 3.1.7. *Let $\mathcal{A}$ be a Coxeter arrangement for a Coxeter group W. Then the posets $\mathrm{Bic}(W)$ and $\mathsf{Quasi}(\mathcal{A})$ are isomorphic. The isomorphism sends $B \in \mathrm{Bic}(W)$ to the quasiregion R with $\mathcal{S}(R) = \{H_t \mid \alpha_t \in B\}$.*

Lemma 3.1.8. *Let $\mathcal{A}$ be a Coxeter arrangement for a Coxeter group W. If R and R' are quasiregions of $\mathcal{A}$ so that $\mathcal{S}(R, R') = \{H_t\}$ where $t \in T$, then $t \cdot R = R'$. Hence if R is a region, then R' is also a region.*

Proof. Assume that $\mathcal{S}(R, R') = \{H_t\}$ for $t \in T$. Indeed, on each full rank 2 subarrangement $\mathcal{A}'$ containing H_t, both R and R' restrict to regions of $\mathcal{A}'$ separated by a single hyperplane. Let W' be the reflection subgroup generated by reflections over hyperplanes in $\mathcal{A}'$. Then the action of W' on the quasiregions of $\mathcal{A}$ commutes with restriction to $\mathcal{A}'$, and the action of t on the regions of $\mathcal{A}'$ swaps pairs of regions with $\mathcal{S}(R, R') = \{H_t\}$. Hence the quasiregions $t \cdot R$ and R' coincide on every full rank 2 subarrangement of $\mathcal{A}$. This fully determines the value of $H^{t \cdot R}$, since each $H \in \mathcal{A}$ is contained in a full rank 2 subarrangement with H_t. We conclude that $t \cdot R = R'$. $\square$

3.2 Suitable orders

In this section we describe the main tool used in our work in [7] to prove Conjectures 3.0.5 and 3.0.6 for affine root systems.

Definition 3.2.1. Let $\mathcal{A}$ be a based hyperplane arrangement with base region R_0 and let $\mathcal{A}'$ be a subarrangement. If R_0' is the region of $\mathcal{A}'$ containing R_0, then a **basic hyperplane** of $\mathcal{A}'$ is a bounding hyperplane of R_0'. We say $\mathcal{A}'$ is a **full subarrangement** if for any $H_1, H_2, H_3 \in \mathcal{A}$ with $H_1, H_2 \in \mathcal{A}'$ and $H_3 \supseteq H_1 \cap H_2$, we have $H_3 \in \mathcal{A}'$. A partial order $\leq$ on $\mathcal{A}$ is **suitable** if it satisfies the following properties:

(1) Every finite subset of $\mathcal{A}$ is contained in a finite $\leq$-order ideal;

(2) Every full rank 2 subarrangement $\mathcal{A}' \subseteq \mathcal{A}$ has two basic hyperplanes H_1, H_2, and if $H \in \mathcal{A}'$ is a non-basic hyperplane, then $H_1, H_2 < H$; and

(3) For any finite $\leq$-order ideal $\mathcal{B}$ and any three hyperplanes $H_1, H_2, H_3 \in \mathcal{A}'$, there exists a full rank 3 subarrangement $\mathcal{A}' \subseteq \mathcal{A}$ containing H_1, H_2, and H_3 so that $\mathcal{A}' \cap \mathcal{B}$ is clean.

If Φ^+ is the set of positive roots in a root system, then a partial order $\leq$ on Φ^+ is **suitable** if the induced ordering on the Coxeter arrangement $\mathcal{A} = \{\alpha^\perp \mid \alpha \in \Phi^+\}$ is suitable.

By an **initial section** of a suitable order on a Coxeter arrangement, we mean a subset $I \subseteq \Phi^+$ so that if $\alpha \in I$ and $\alpha'^\perp < \alpha^\perp$, then $\alpha' \in I$. We note that reflection orders are never suitable orders (except for Coxeter groups of the form $A_1 \times \cdots \times A_1$), despite the fact that the language surrounding them is similar.

The following is a main result of [7]. The **root poset** on a root system Φ with base Π puts $\alpha \leq \beta$ if $\beta - \alpha \in \mathrm{cone}(\Pi)$.

Theorem 3.2.2. *Let Φ be a finite crystallographic or untwisted affine root system. Then the root poset is a suitable order on Φ^+.*

The other main result of [7] is the motivation for considering suitable orders. The **closure** of a set $U \subseteq \Phi^+$ is the minimal closed set $\overline{U} \supseteq U$.

Theorem 3.2.3. *Let W be a Coxeter group with a Coxeter arrangement $\mathcal{A}$ having a suitable order. Then $\mathrm{Bic}(W)$ satisfies Dyer's Conjectures A and B. The join of a collection of biclosed sets $\{B_i\}_{i \in I}$ is*

$$\bigvee_{i \in I} B_i = \overline{\bigcup_{i \in I} B_i}.$$

We record another lemma from [7] that will be useful.

Lemma 3.2.4. *Let Φ^+ be a positive root system with a suitable order. If $I \subseteq \Phi^+$ is an initial section, then for every biclosed set $B' \in \mathrm{Bic}(I)$ viewed as a subset of Φ^+, the closure $\overline{B'}$ is in $\mathrm{Bic}(\Phi^+)$. Furthermore, $\overline{B'} \cap I = B'$.*

Corollary 3.2.5. *Let Φ^+ be a positive root system with a suitable order. If $I \subseteq \Phi^+$ is an initial section, then the restriction map $\mathrm{Bic}(\Phi^+) \to \mathrm{Bic}(I)$ is a complete lattice quotient.*

Proof. We first show that for each $B \in \mathrm{Bic}(I)$ there exists $B^\downarrow, B^\uparrow \in \mathrm{Bic}(\Phi^+)$ so that the set of biclosed sets in $\mathrm{Bic}(\Phi^+)$ restricting to B is exactly $[B^\downarrow, B^\uparrow]_{\mathrm{Bic}(\Phi^+)}$. By Lemma 3.2.4, there is a unique minimal biclosed set $B^\downarrow = \overline{B}$ which restricts to B. Applying Lemma 3.2.4 now to $I \setminus B$, we see there is a unique minimal biclosed set restricting to $I \setminus B$, which we could denote by $I \setminus B^{\uparrow}$. Hence every biclosed set in Φ^+ restricting to B is in the interval $[B^{\downarrow}, B^{\uparrow}]$.

Furthermore, the map $\mathrm{Bic}(\Phi^+) \to \mathrm{Bic}(\Phi^+)$ defined by $B \mapsto (B \cap I)^{\downarrow} = \overline{B \cap I}$ is order-preserving, as is the map $B \mapsto (B \cap I)^{\uparrow}$. Hence by Proposition 2.4.2, $\mathrm{Bic}(\Phi^+) \to \mathrm{Bic}(I)$ is a complete lattice quotient. □

Theorem 3.2.6. *If* Φ^+ *is a positive root system with a suitable ordering, then* $\mathrm{Bic}(\Phi^+)$ *is a profinite lattice.*

Proof. Let $B_1, B_2 \in \mathrm{Bic}(\Phi^+)$ be distinct biclosed sets. Then there is some $\alpha \in B_1 \Delta B_2$. Let I be a finite initial section of the suitable order containing α. By Corollary 3.2.5, the restriction map $\mathrm{Bic}(\Phi^+) \to \mathrm{Bic}(I)$ is a complete lattice quotient. Evidently $B_1 \cap I$ and $B_2 \cap I$ are distinct, since one contains α and the other does not. Furthermore, $\mathrm{Bic}(I)$ is a finite lattice. Hence $\mathrm{Bic}(\Phi^+)$ is profinite. □

3.3 Semidistributivity and polygonality for $\mathrm{Bic}(W)$

In the theory of the poset of regions of a finite hyperplane arrangement, there is a close relationship between semidistributivity, a lattice-theoretic property, and *bisimpliciality* (or *tightness*), a geometric property.

Definition 3.3.1. We say that a finite based hyperplane arrangement $\mathcal{A}$ is **bisimplicial** if, for any region R and any lower walls (equivalently, any upper walls) H_1, H_2 of R, the intersection $H_1 \cap H_2$ is incident to R. We say that a suitable order $\leq$ on a based hyperplane arrangement $\mathcal{A}$ is **semidistributivitable** if, for any finite $\leq$-order ideal $\mathcal{B}$ and any three hyperplanes $H_1, H_2, H_3 \in \mathcal{B}$, there is a full rank 3 subarrangement $\mathcal{A}' \subseteq \mathcal{A}$ containing H_1, H_2, and H_3 so that $\mathcal{A} \cap \mathcal{B}$ is clean and bisimplicial.

The relationship between bisimplicial arrangements and semidistributive lattices is the following [45, Theorem 9-3.8]. Write $\mathsf{Regions}(\mathcal{A})$ for the collection of regions of a finite based arrangement $\mathcal{A}$ with base region R_0, partially ordered so that $R_1 \leq R_2$ if $\mathcal{S}(R_0, R_1) \subseteq \mathcal{S}(R_0, R_2)$.

Proposition 3.3.2. *A finite based hyperplane arrangement $\mathcal{A}$ is bisimplicial if and only if* $\mathsf{Regions}(\mathcal{A})$ *is a semidistributive lattice.*

Recall that a profinite lattice L is **pro-polygonal** if for every cofinite congruence $\equiv$ on L, the finite lattice $L/{\equiv}$ is polygonal.

Theorem 3.3.3. *Let $\mathcal{A}$ be a hyperplane arrangement with a semidistributivitable suitable order. Then* $\mathsf{Quasi}(\mathcal{A})$ *is a semidistributive pro-polygonal lattice.*

Proof. By Proposition 2.4.21, Lemma 2.4.30, and Corollary 3.2.5 it is enough to show that for any finite suitable order ideal I, the finite lattice $\mathrm{Bic}(I)$ is semidistributive and polygonal. Let $\mathcal{B}$ be the hyperplane arrangement dual to I. Let $X_1, X_2 \in \mathrm{Bic}(I)$ cover a biclosed set $X \in \mathrm{Bic}(I)$. By [7, Lemma 3.5], there are roots $\beta_1, \beta_2 \in I$ so that $X_1 = X \cup \{\beta_1\}$ and $X_2 = X \cup \{\beta_2\}$. Let Φ' be the full rank 2 subsystem of Φ spanned by β_1, β_2, and $I' = \Phi' \cap I$. We claim $Y := X \cup I'$ is biclosed. Since $Y \subseteq \overline{X_1 \cup X_2}$, this will imply that Y is the join of X_1 and X_2 in $\mathrm{Bic}(I)$. If Y is not biclosed, then there exist $\alpha, \beta, \gamma \in I$ so that either $\alpha, \beta \in Y$ and $\gamma \notin Y$, or $\alpha, \beta \notin Y$ and $\gamma \in Y$. Since X is biclosed, at least one of α, β, γ is in I'. In either case, semidistributivitability implies that there is a full rank 3 subarrangement $\mathcal{A}'$ containing $\alpha^\perp, \beta^\perp, \beta_1^\perp$ (which will then necessarily contain $\gamma^\perp$ and $\beta_2^\perp$ as well) so that $\mathcal{A}' \cap \mathcal{B}$ is clean and bisimplicial. Then there is a region R of $\mathcal{A}' \cap \mathcal{B}$ corresponding to the restriction of X, and $\beta_1^\perp, \beta_2^\perp$ are upper walls of R. By bisimpliciality, there is a region R' corresponding to the restriction of Y. But that means that the restriction of Y is biclosed, contradicting the fact that $\alpha^\perp, \beta^\perp, \gamma^\perp \in \mathcal{A}' \cap \mathcal{B}$. Hence Y is biclosed.

Now, if X_1, X_2, X are as above, then we have shown that $X_1 \vee X_2 = X \cup I'$ (using the notation above). In particular, the interval $[X, X_1 \vee X_2]$ is isomorphic to $\mathrm{Bic}(I')$, and hence is a polygon. Furthermore, if $Z \in \mathrm{Bic}(I)$ is such that $Z \wedge X_1 = Z \wedge X_2$, then neither β_1 nor β_2 is in Z. Hence $Z \wedge X_1 = Z \wedge X_2 = Z \wedge X$. Furthermore, neither β_1 nor β_2 is in $Z \wedge (X_1 \vee X_2) = Z \wedge Y$, which implies (since all non-empty biclosed subsets of I' contain β_1

or β_2) that $Z \wedge Y \subseteq (Z \wedge Y) \setminus I_2 \subseteq Z \wedge X$. Evidently, this implies $Z \wedge Y = Z \wedge$. We have shown that whenever X_1, X_2 cover a common element in $\mathrm{Bic}(I)$ and $Z \wedge X_1 = Z \wedge X_2$, then $Z \wedge (X_1 \vee X_2) = Z \wedge X_1$. By [45, Lemma 9-2.6], $\mathrm{Bic}(I)$ is a meet semidistributive lattice.

By applying the complementation operator on $\mathrm{Bic}(I)$, we deduce that $\mathrm{Bic}(I)$ is join semidistributive and has the property that whenever X_1, X_2 are covered by a common element, then $[X_1 \wedge X_2, X_1 \vee X_2]$ is a polygon. Hence $\mathrm{Bic}(I)$ is a semidistributive polygonal lattice. □

We make the following conjecture, which would imply that $\mathrm{Bic}(W)$ is semidistributive and pro-polygonal for any Coxeter group W.

Conjecture 3.3.4. *Every Coxeter arrangement has a semidistributivitable order.*

At the moment, it is not even known that there is a suitable order for Coxeter groups that are not finite or affine. If W is finite, than any suitable order for W will be semidistributivitable. We now show that affine Coxeter groups also have semidistributivitable suitable orders, from which it follows that Conjecture 3.0.7 holds for affine Coxeter groups.

Lemma 3.3.5. *Let $I' \subseteq I \subseteq \Phi^+$ be finite suitable order ideals so that the hyperplane arrangement $\mathcal{A} = \{\alpha^\perp \mid \alpha \in I\}$ is clean and bisimplicial. Then the hyperplane arrangement $\mathcal{A}' = \{\alpha^\perp \mid \alpha \in I\}$ is bisimplicial.*

Proof. From Corollary 3.2.5, we have lattice quotient maps $\mathrm{Bic}(\Phi^+) \twoheadrightarrow \mathrm{Bic}(I)$ and $\mathrm{Bic}(\Phi^+) \twoheadrightarrow \mathrm{Bic}(I')$. Furthermore, the function $\eta : \mathrm{Bic}(I) \to \mathrm{Bic}(I')$ given by $B \mapsto B \cap I'$ fits in the commutative diagram

$$\begin{tikzcd} & \mathrm{Bic}(I) \arrow[dr, "\eta"] & \\ \mathrm{Bic}(\Phi^+) \arrow[ur] \arrow[rr] & & \mathrm{Bic}(I') \end{tikzcd}.$$

Since the other two maps in the diagram are surjective complete lattice homomorphisms, so is η. If $\mathcal{A}$ is clean, then so is $\mathcal{A}'$ (by lifting $B' \in \mathrm{Bic}(I')$ to some $B \in \mathrm{Bic}(I)$ and taking a separating functional for B). Since $\mathcal{A}$ is bisimplicial, we deduce by Theorem 9-3.8 of [45] that $\mathrm{Bic}(I)$ is semidistributive and by Proposition 9-5.26 that $\mathrm{Bic}(I')$ is semidistributive. Hence by Theorem 9-3.8, $\mathcal{A}'$ is bisimplicial. □

Remark 3.3.6. In fact, if $\mathcal{A}$ is any finite bisimplicial arrangement and $\mathcal{A}'$ is an order ideal in some partial order satisfying Definition 3.2.1(2), then $\mathcal{A}'$ is also bisimplicial, by [45, Theorem 9-3.8 and Propositions 9-5.26 and 9-8.6].

Theorem 3.3.7. *Let Φ be an untwisted affine root system. Then the root poset on Φ^+ is semidistributivitable.*

Proof. It is enough to show that if Φ is a rank 3 subsystem of an untwisted affine root system and I is a finite root poset order ideal of Φ^+, then $\mathcal{A}_I = \{\alpha^\perp \mid \alpha \in I\}$ is a bisimplicial arrangement. The root subsystems of untwisted affine root systems are products of finite and affine root systems. Taking products preserves bisimpliciality, and rank 1 and 2 arrangements are always bisimplicial, so we can reduce to the case where Φ is irreducible. In this case Φ is either finite or untwisted affine. If Φ is finite and $I = \Phi$ then it is well-known that $\mathcal{A}_I$ is bisimplicial (in fact, simplicial). Hence by Lemma 3.3.5 $\mathcal{A}_I$ is bisimplicial for any suitable order ideal of Φ.

Now assume Φ is untwisted affine, so that $\Phi = \Phi_0 + \mathbb{Z}\delta$ for an irreducible rank 2 finite root system Φ_0 (hence Φ_0 is one of A_2, C_2, or G_2). Then Φ^+ is the increasing union of the *m-Shi arrangements* $\mathcal{A}_m = \{(\alpha + k\delta)^\perp \mid \alpha \in \Phi_0,\ k \in \{-m, -m+1, \ldots, m-1\}\}$, each of which is a root poset ideal. By Lemma 3.3.5, it is enough to check that each $\mathcal{A}_m$ is bisimplicial; equivalently, that if R is a region which is not the base region R_0 or its negation $-R_0$, then R has at most two lower walls and at most two upper walls. See Figure 3.1. Each region of $\mathcal{A}_m$ either intersects the plane $\delta^\perp$ or it does not. The regions R intersecting $\delta^\perp$ are the unbounded regions in Figure 3.1 without parallel walls; there are exactly $|\Phi_0|$ of them and each has two lower walls and two upper walls. If R is a region not intersecting $\delta^\perp$, then either R or $-R$ is contained in the Tits cone; without loss of generality, R is in the Tits cone. If R has parallel walls, then R has at most 3 walls hence is simplicial.

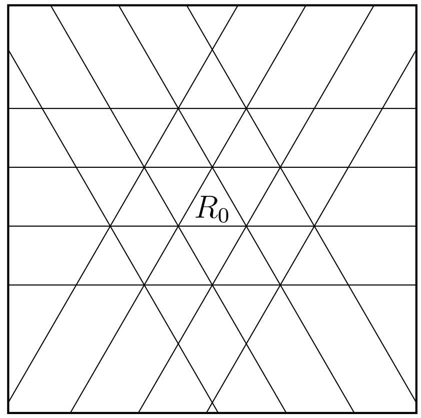


Figure 3.1: The type $\widetilde{A}_2$ 2-Shi arrangement $\mathcal{A}_2$. Note that $\mathcal{A}_2$ is a rank 3 hyperplane arrangement, and what is depicted is a two-dimensional affine slice through the Tits cone.

Otherwise, R is bounded within the Tits cone. This means that any region of the full Coxeter arrangement $\mathcal{A}$ contained in R must be bounded within the Tits cone. By Lemma 3.2.4, there is a unique minimal quasiregion $R^{\downarrow} \in \mathsf{Quasi}(\mathcal{A})$ contained in R and a unique maximal quasiregion $R^{\uparrow} \in \mathsf{Quasi}(\mathcal{A})$ contained in R. By Proposition 3.1.6, $\mathcal{A}$ is clean, so $R^{\downarrow}$ and $R^{\uparrow}$ are regions. Regions which are bounded within the Tits cone are exactly those which are in the W orbit of R_0. In particular, each such region is simplicial. We claim that the upper walls of R in $\mathcal{A}_m$ are exactly the upper walls of $R^{\uparrow}$ in $\mathcal{A}$, and the lower walls of R in $\mathcal{A}_m$ are exactly the lower walls of $R^{\downarrow}$ in $\mathcal{A}$. In particular, R is either R_0 or has at most two upper and two lower walls. We address the claim about upper walls, the other is similar. First, the upper walls of $R^{\uparrow}$ are evidently included among the upper walls of R, or else we can find a larger region of $\mathcal{A}$ contained in R. For the converse, let H be an upper wall of R which is not an upper wall of $R^{\uparrow}$. There is some maximal region R_H of $\mathcal{A}$ contained in R among those regions with H as an upper wall. Since $R_H \neq R^{\uparrow}$, there is some region $R' \in \mathcal{A}$ so that $R' \gtrdot R_H$ and R' is also contained in R. If H' is the hyperplane separating R' and R_H, then H, H' are the basic hyperplanes of a full rank 2 subarrangement $\mathcal{A}' \subseteq \mathcal{A}$. Since $\mathcal{A}_m$ is a suitable order ideal and $H' \notin \mathcal{A}_m$, by Definition 3.2.1(2) we have that $\mathcal{A}_m \cap \mathcal{A}' = \{H\}$. It follows that the region R'' of $\mathcal{A}$ with $\mathcal{S}(R'', R_H) = \mathcal{A}' \setminus \{H\}$ is also contained in R. But this contradicts the maximality of R_H, so we are done. $\square$

Corollary 3.3.8. *Let W be an affine Coxeter group. Then* $\mathrm{Bic}(W)$ *is semidistributive and pro-polygonal.*

Let I be a finite subset of Φ^+ and let $\mathcal{C}$ be a maximal chain in $\mathrm{Bic}(I)$. If Φ' is a full rank 2 subsystem of Φ^+ so that there are $\{B_1 \lessdot B_2 \lessdot \cdots \lessdot B_k\} \subseteq \mathcal{C}$ with $B_k \setminus B_1 = I \cap \Phi'^+$, then the **rank 2 flip** of $\mathcal{C}$ is the maximal chain

$$(\mathcal{C} \setminus \{B_2, \ldots, B_{k-1}\}) \cup \{B_1 \cup (B_k \setminus B_{k-1}), B_1 \cup (B_k \setminus B_{k-2}), \ldots, B_k\}.$$

Theorem 3.3.9. *Let Φ^+ be a positive root system with a semidistributivitable suitable order. Let $I \subseteq \Phi^+$ be a finite initial section of Φ^+. Then any two maximal chains in* $\mathrm{Bic}(I)$ *are connected by a sequence of rank 2 flips.*

Proof. By the proof of Theorem 3.3.3, the lattice $\mathrm{Bic}(I)$ is polygonal and the polygons correspond to intervals of the form $\{X \cup B \mid B \text{ is a biclosed subset of } I'\}$ for X a biclosed set and $I' = I \cap \Phi'$ for some full rank 2 subsystem Φ'. In particular, rank 2 flips coincide with the *polygon moves* described for polygonal lattices in [45]. Hence by [45, Lemma 9-6.3], any two maximal chains of $\mathrm{Bic}(I)$ are related by a sequence of rank 2 flips. □

Remark 3.3.10. This can be seen as a generalization of *Matsumoto's theorem*, which says that any two reduced words for an element of a Coxeter group are related by braid moves. When Φ is a finite root system and $I = \Phi^+$, maximal chains in $\mathrm{Bic}(I)$ can be identified with reduced expressions for the longest element of the Weyl group W, and rank 2 moves correspond to applying braid relations to the expression.

3.4 Type $\widetilde{A}_{n-1}$

Throughout this section we fix a natural number n. We record here some explicit combinatorics describing $\mathrm{Bic}(W)$ when W is of type $\widetilde{A}_{n-1}$, which will be used in Section 4.5.5. More detail on this material can be found in [4].

3.4.1 The affine symmetric group and its root system

Here we describe the Coxeter group of type $\widetilde{A}_{n-1}$.

Definition 3.4.1. The **affine symmetric group** $\widetilde{S}_n$ is the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ satisfying:

(a) $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$, and

(b) $\displaystyle\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i$.

To define the root datum for $\widetilde{S}_n$, let $\mathbb{R}^{\mathbb{Z}}$ be the vector space with basis $\{\mathbf{e}_i\}_{i\in\mathbb{Z}}$. Define V to be the quotient vector space by the relations $\mathbf{e}_{i+n} - \mathbf{e}_i = \mathbf{e}_{j+n} - \mathbf{e}_j$ for $i, j \in \mathbb{Z}$. Let $\widetilde{\mathbf{e}}_i$ be the image of $\mathbf{e}_i$ in $\widetilde{V}$. For an integer i, set $\alpha_i \coloneqq \widetilde{\mathbf{e}}_i - \widetilde{\mathbf{e}}_{i+1}$. For integers $i \leq j$, set $\alpha_{i,j} \coloneqq \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, and for integers $i \geq j$, set $\alpha_{i,j} \coloneqq -\alpha_{j,i}$. Define a bilinear form $(-,-)$ on V so that $(\widetilde{\mathbf{e}}_i, \widetilde{\mathbf{e}}_j) = \begin{cases} 1 & \text{if } i \equiv j \bmod n \\ 0 & \text{otherwise} \end{cases}$. Set $\Pi = \{\alpha_0, \ldots, \alpha_{n-1}\}$. Then there is a symmetrizable crystallographic simplicial root datum $(V, \Pi, V^*, \Pi^\vee)$ with root system Φ. The affine symmetric group acts on V by $w\widetilde{\mathbf{e}}_i = \widetilde{\mathbf{e}}_{w(i)}$, and this realizes W as the Weyl group of Φ.

The primitive imaginary root is $\delta \coloneqq \alpha_{0,n} \in \Delta^+_{\mathrm{im}}$. Every imaginary root in the Kac–Moody root system Δ is an integer multiple of δ.

3.4.2 TITOs

Definition 3.4.2. A **translation-invariant total order** (**TITO**) is a total order $(\prec)$ of the integers such that for all $a, b \in \mathbb{Z}$, we have $a \prec b$ if and only if $a + n \prec b + n$. We write TTot_n for the set of TITOs.

Example 3.4.3. Here are three examples of TITOs with $n = 4$.

$$\cdots \prec -2 \prec -3 \prec 0 \prec -1 \prec 2 \prec 1 \prec 4 \prec 3 \prec 6 \prec 5 \prec 8 \prec 7 \prec \cdots \tag{3.4.1}$$

$$\cdots \prec 0 \prec 1 \prec 4 \prec 5 \prec 8 \prec 9 \prec \cdots \prec 10 \prec 11 \prec 6 \prec 7 \prec 2 \prec 3 \prec \cdots \tag{3.4.2}$$

$$\cdots \prec 0 \prec 4 \prec 8 \prec 12 \prec \cdots \prec -2 \prec -3 \prec -1 \prec 2 \prec 1 \prec 3 \prec 6 \prec 5 \prec 7 \prec \cdots . \tag{3.4.3}$$

To abbreviate this data, we use **window notation** to encode a TITO. The key observation is that a TITO $\prec$ decomposes into **blocks**, which are subsets of $\mathbb{Z}$ on which $\prec$ restricts to an ordering which is order-isomorphic to $\mathbb{Z}$.

Definition 3.4.4. Let $(\prec)$ be a TITO. A **block** of $(\prec)$ is a nonempty order-convex subset I with the following properties:

- The ordering of I by $(\prec)$ has no minimal or maximal element, and
- For any $a, c \in I$, the interval $\{b \in I \mid a \prec b \prec c\}$ is finite.

The **size** of a block I is the number of residue classes modulo n appearing in I. We say that I is a **waxing** block if $x \prec x+n$ for all $x \in I$. We say that I is a **waning** block if $x+n \prec x$ for all $x \in I$.

For example, the TITO (3.4.2) has two blocks: the first has the integers congruent to 0 or 1 modulo 4, and the second contains the rest. Each block has its own *window*: if a block contains k residue classes, then a window for it consists of k consecutive integers from the block. Furthermore, if the block is waning then we underline the window, and if the block is waxing then we do not underline the window. For example, the window notations for (3.4.1), (3.4.2), and (3.4.3) are $[2, 1, 4, 3]$, $[0, 1][\underline{2, 3}]$, and $[0][2, 1, 3]$, respectively.

Definition 3.4.5. Let $\prec$ be a TITO. An **inversion** of $\prec$ is a positive real root $\alpha_{i,j}$ so that $j \prec i$. Thus

$$\operatorname{inv}(\prec) := \{\alpha_{i,j} \mid i < j,\ i \not\equiv j \bmod n,\ j \prec i\}$$

is the **inversion set** of $\prec$.

The following was shown in [8].

Proposition 3.4.6. *A set of positive roots $B \subseteq \Phi^+$ is biclosed if and only if there is a TITO $(\prec)$ so that* $\mathrm{inv}(\prec) = B$. *Two TITOs have the same inversion set if and only if they differ by reversing blocks of size 1.*

3.4.3 Arcs

In this section we give a parametrization of the complete join-irreducibles of $\mathrm{Bic}(\widetilde{S}_n)$ using *arc data*.

Definition 3.4.7. An **arc tuple** is a tuple (a, b, L, R), where $a, b \in \mathbb{Z}$ have $a < b$ and $L \sqcup R$ is a partition of $\{a + 1, \ldots, b - 1\}$ into two subsets. An **arc datum** is an equivalence class of arc tuples under the relation $(a, b, L, R) \sim (a + n, b + n, L + n, R + n)$. The arc datum containing (a, b, L, R) is denoted $\boldsymbol{(}a, b, L, R\boldsymbol{)}$. We say the arc datum is **real** if $a \not\equiv b \bmod n$ and **imaginary** if $a \equiv b \bmod n$.

We say that arc tuples (a, b, L, R) and (a', b', L', R') **cross** if either $a = a'$, $b = b'$, or the sets $(L \cap R') \cup (\{a, b\} \cap R') \cup (L \cap \{a', b'\})$ and $(R \cap L') \cup (\{a, b\} \cap L') \cup (R \cap \{a', b'\})$ are both nonempty. The arc data $\boldsymbol{(}a, b, L, R\boldsymbol{)}$ and $\boldsymbol{(}a', b', L', R'\boldsymbol{)}$ **cross** if there exist $j, k \in \mathbb{Z}$ so that the arcs $(a + jn, b + jn, L + jn, R + jn)$ and $(a' + kn, b' + kn, L' + kn, R' + kn)$ cross. We say an arc datum is **non-crossing** if it does not cross itself. We may depict arc tuples as in Figure 3.2, using an arc traveling from a to b passing over the elements of L and under the elements of R. We may depict arc data as in Figure 3.3, by either drawing the full collection of arc tuples in the equivalence class, or by drawing a single arc tuple on a circle. In either picture, the non-crossing condition translates to the arc not intersecting itself.

Remark 3.4.8. The diagram showing the arc wrapped on a circle can be interpreted as living in the target of a Lefschetz fibration from a 2-dimensional variety we will discuss later. The unwrapped diagram lives in the universal cover of the target. A real non-crossing arc datum, drawn as an arc wrapped on the circle, is the image of a Lagrangian sphere under the Lefschetz fibration. An imaginary noncrossing arc datum is the image of a singular Lagrangian torus. We will also see in Section 4.5.5 how to turn an arc datum into a module

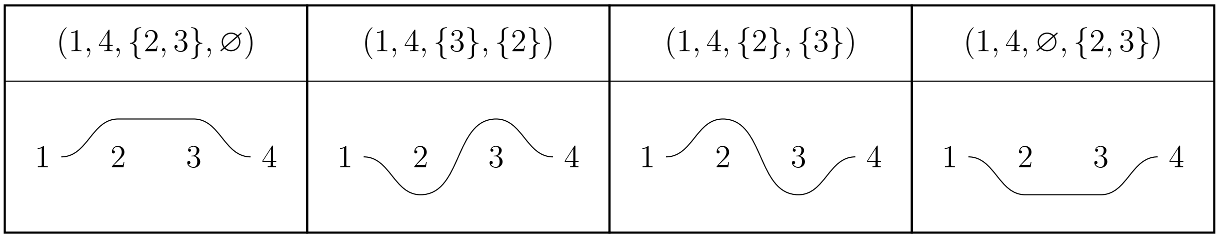


Figure 3.2: The arc tuples with initial value 1 and terminal value 4.

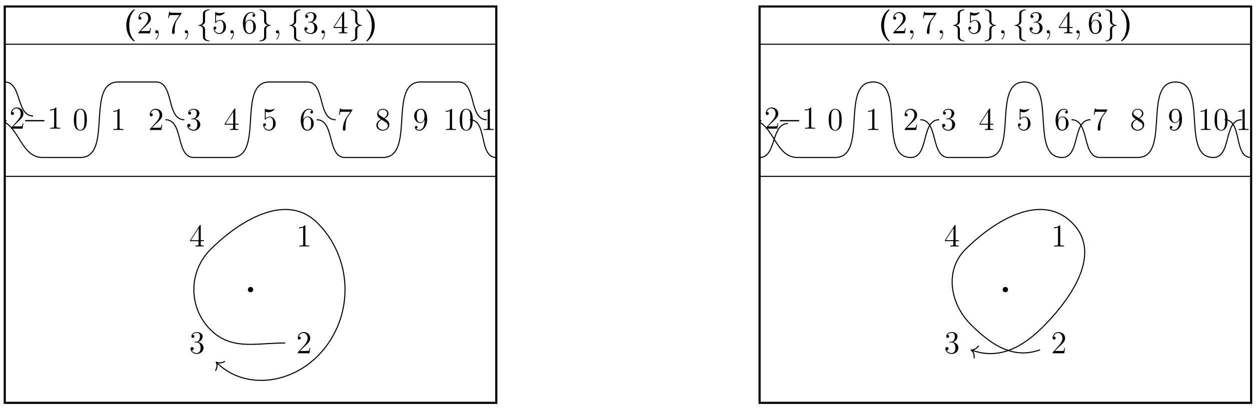


Figure 3.3: On the left, a non-crossing arc datum drawn in two ways: "unwrapped" and "wrapped". On the right, an arc datum with a crossing.

for a preprojective algebra. Then the noncrossing condition is equivalent to the module being a *brick*.

Recall the definition of a complete join-irreducible (cJI) from Definition 2.4.13. Given a cJI $J \in \mathrm{Bic}(\widetilde{S}_n)$, we define an arc datum $\mathfrak{a}(J)$ as follows. Let $(\prec)$ be the unique TITO with no waning blocks of size one so that $\mathrm{inv}(\prec) = J$. (Equivalently, $(\prec)$ is the minimal TITO such that $\mathrm{inv}(\prec) = J$.) Then we set $\mathfrak{a}(J) := (a, b, L, R)$, where $b \prec a$ are consecutive entries in $(\prec)$ which are inverted, and $L = \{z \in \mathbb{Z} \mid a < z < b,\ z \prec b\}$, $R = \{z \in \mathbb{Z} \mid a < z < b,\ a \prec z\}$.

Proposition 3.4.9 ([4])**.** *The map $J \mapsto \mathfrak{a}(J)$ is a bijection between $\mathsf{JIrr}^c(\mathrm{Bic}(\widetilde{S}_n))$ and real noncrossing arc data.*

It will be useful to describe the cJIs explicitly. Let $\mathfrak{a} = (a, b, L, R)$ be a real noncrossing arc datum. Write $J_{\mathfrak{a}}$ for the unique cJI with $\mathfrak{a}(J_{\mathfrak{a}}) = \mathfrak{a}$.

Case 1: If $b - a < n$, then with $L = \{\ell_1 < \cdots < \ell_i\}$ and $R = \{r_1 < \cdots < r_j\}$

$$J_{\mathfrak{a}} = \mathrm{inv}([\ell_1, \ldots, \ell_i, b, a, r_1, \ldots, r_j, b+1, b+2, \ldots, n+a-1]).$$

Case 2: If $b - a > n$ and $a + n \in R$, then for an integer x let $\mathcal{C}(x)$ denote the maximum element of $L \cup \mathbb{Z}_{<a}$ which is congruent to x mod n. Let $c_1 < \cdots < c_{n-2}$ be the elements of $\{\mathcal{C}(x) \mid x \not\equiv a, b \bmod n\}$ in sorted order. Then

$$J_{\mathfrak{a}} = \mathrm{inv}([c_1, \ldots, c_{n-2}, b, a]).$$

Case 3: If $b - a > n$ and $a + n \in L$, then for an integer x let L_x be the elements of L congruent to x mod n and let R_x be the elements of R congruent to x mod n. Let $\mathcal{L}$ be the set of integers x so that $L_x \neq \varnothing$ and $R_x = \varnothing$, let $\mathcal{C}$ be the set of integers x so that $L_x, R_x \neq \varnothing$, and let $\mathcal{R}$ be the set of integers x so that $R_x \neq \varnothing$ and $L_x = \varnothing$. Let $\ell_{i_1} < \cdots < \ell_{i_j}$ be the elements of $\{\max L_x \mid x \in \mathcal{L},\ x \not\equiv a \bmod n\}$, and let $r_{i_1} < \cdots < r_{i_k}$ be the elements of $\{\min R_x \mid x \in \mathcal{R},\ x \not\equiv b \bmod n\}$. Let $c_1 < \cdots < c_r$ be the elements of $\{\max L_x \mid x \in \mathcal{C}\}$. Then

$$J_{\mathfrak{a}} = \mathrm{inv}([\ell_{i_1}, \ldots, \ell_{i_j}][\underline{c_1, \ldots, c_r, b, a}][r_{i_1}, \ldots, r_{i_k}]).$$

Chapter 4

Calabi-Yau categories

In this chapter we will apply biclosed sets to the representation theory of preprojective algebras and the related problem of studying t-structures on Calabi-Yau categories. The material presented here benefited from helpful conversations with Denis Auroux, Andrew Burke, Merrick Cai, Keeley Hoek, David Speyer, and Hugh Thomas.

4.1 Quivers and algebras

In this section we discuss *Calabi-Yau categories*, which are the setting for our main results of the chapter. We also discuss some important examples from algebra and geometry. Fix k an algebraically closed field. All of our categories are k-linear, meaning enriched over the category of k-vector spaces.

4.1.1 Quiver representations

A **quiver** Q consists of a set of vertices Q_0 and a set of arrows Q_1 along with functions $s, t : Q_1 \to Q_0$. The vertex $s(a)$ is called the **source** of a and the vertex $t(a)$ is called the **target** of a. A **representation** M of Q is a collection of vector spaces $(M_v)_{v\in Q_0}$ along with linear maps $L_a : M_{s(a)} \to M_{t(a)}$ for each $a \in Q_1$. Let V be the real vector space with basis $\{\alpha_v\}_{v\in Q_0}$. Then the **dimension vector** of a representation M is $\underline{\dim}(M) := \sum_{v\in Q_0} \dim(M_v)\alpha_v$.

Example 4.1.1. Let Q be the quiver $1 \xrightarrow{a} 2$. Then a representation for Q consists of vector spaces V_1, V_2 and a linear map $a : V_1 \to V_2$. For example, we could take $V_1 = k$, $V_2 = k$, and $a = v \mapsto 2v$, which we could depict by

$$k \xrightarrow{2} k \, .$$

A **path** in Q of length k is a sequence of arrows $\mathbf{a} = (a_1, \ldots, a_k)$ so that $s(a_i) = t(a_{i+1})$ for all i. We say the source of $\mathbf{a}$ is $s(a_k)$ and the target of $\mathbf{a}$ is $t(a_1)$. For each vertex v, we also include the empty path (v) centered at v, which has source and target v and length 0. The **path algebra** of Q is the k-algebra with basis given by the paths of Q and multiplication sending

$$\mathbf{a} \cdot \mathbf{b} = \begin{cases} (a_1, \ldots, a_k, b_1, \ldots, b_k) & \text{if } s(\mathbf{a}) = t(\mathbf{b}) \\ 0 & \text{if } s(\mathbf{a}) \neq t(\mathbf{b}) \end{cases} .$$

The identity element of $k[Q]$ is $\sum_{v \in Q_0} (v)$. Representations of Q are equivalent to (left) modules for $k[Q]$. Explicitly, a $k[Q]$ module M gives a representation with $M_v = (v)M$ for each $v \in Q_0$ and $L_a = (a)$ for each $a \in Q_1$.

We say that an algebra A is of **finite representation type** if there are finitely many isomorphism classes of finite-dimensional indecomposable A-modules. We say that a quiver Q is a **Dynkin quiver** if it is isomorphic as an undirected graph to a Dynkin diagram of type A, D, or E (see Figure 2.4).

Theorem 4.1.2 (Gabriel)**.** *Let Q be a connected quiver. Then $k[Q]$ is of finite representation type if and only if Q is a Dynkin quiver.*

Gabriel's theorem admits a strengthening due to Kac, describing the dimension vectors of indecomposable $k[Q]$-modules in terms of roots. Fix a quiver Q with no loops (arrows a with $s(a) = t(a)$). Then Q defines a symmetric Cartan matrix $(A_{ij})_{i,j \in Q_0}$, where for $i \neq j$, the value of A_{ij} is the number of arrows with source and target both in the set $\{i, j\}$. Let Δ_Q denote the simplicial Kac–Moody root system of A_{ij} (as in Section 2.3.4).

Theorem 4.1.3 (Kac [35])**.** *Let Q be a quiver with no loops. If M is an indecomposable representation of Q, then $\underline{\dim}(M)$ is in Δ_Q^+. If $\alpha \in \Delta^+_{\mathrm{re},Q}$, then there exists a unique indecomposable representation of Q with dimension vector α. If $\alpha \in \Delta^+_{\mathrm{im},Q}$, then there exist infinitely many indecomposable representations of Q with dimension vector α.*

We say that a quiver Q is an **affine quiver** if it is isomorphic as an undirected graph to an extended Dynkin diagram of type $\widetilde{A}$, $\widetilde{D}$, or $\widetilde{E}$ (see Figure 2.5); equivalently, Q is connected and (A_{ij}) is positive semi-definite. As a special case of Kac's theorem, the indecomposable representations of an affine quiver consist of a single representation for each real root, and finitely many one-parameter families of representations of dimension $k\delta$ for each $k \in \mathbb{N}$.

4.1.2 Preprojective algebras

Definition 4.1.4. Let Q be a quiver without loops. The **doubled quiver** $\overline{Q}$ has vertices $\overline{Q}_0 = Q_0$ and arrows $\overline{Q}_1 = Q_1 \sqcup Q_1^*$, where elements in Q_1^* are formal copies of elements in Q_1, so that for each $a \in Q_1$ the arrow $a^* \in Q_1^*$ has $s(a^*) = t(a)$ and $t(a^*) = s(a)$. The **preprojective algebra** is the quotient of the path algebra $k[\overline{Q}]$ defined by

$$\Pi_Q \coloneqq k[\overline{Q}]/\left(\sum_{a \in Q_1} (aa^* - a^*a)\right).$$

We are primarily interested in categories of (left) modules for Π_Q. Note that since Π_Q is a quotient of $k[\overline{Q}]$, we will freely think of modules for Π_Q as modules M for $k[\overline{Q}]$ satisfying the property that $\left(\sum_{a\in Q_1}(aa^* - a^*a)\right) M = 0$. Hence modules for Π_Q may be described as certain quiver representations for the doubled quiver $\overline{Q}$. In particular, each Π_Q-module M decomposes as $M = \bigoplus_{v\in Q_0} M_v$, where M_v is the vector space associated to v in the representation of $\overline{Q}$. Let V be the real vector space with basis $\{\alpha_v\}_{v\in Q_0}$. The **dimension vector** of a module M is $\underline{\dim}\, M \coloneqq \sum_{v\in Q_0} \dim(M_v)\alpha_v$. For each vertex $v \in Q_0$, there is a unique Π_Q-module S_v up to isomorphism with $\underline{\dim}(S_v) = \alpha_v$, which is necessarily simple.

We say a module M is **nilpotent** if it is finite dimensional and there is some $k > 0$ so that any path $\mathbf{a}$ of length $\geq k$ annihilates M. Equivalently, M has a finite-length composition series with composition factors in the set $\{S_1, \ldots, S_n\}$.

4.2 Calabi-Yau categories

4.2.1 2-Calabi-Yau categories and spherical objects

Recall that a triangulated category is a category $\mathcal{C}$ equipped with the data of a shift functor $[1] : \mathcal{C} \to \mathcal{C}$ and a collection of pairs of composable morphisms, denoted $A \to B \to C \to$ and called *exact triangles*, subject to some conditions.

Definition 4.2.1. Let $\mathcal{C}$ be a triangulated category with shift functor [1]. If there are isomorphisms

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(B, A[2])^*$$

which are functorial in both variables, then $\mathcal{C}$ is a **2-Calabi-Yau category**.

By *Calabi-Yau category*, we will always mean 2-Calabi-Yau category.

Definition 4.2.2. Let $\mathcal{C}$ be a Calabi-Yau category. A **spherical object** of $\mathcal{C}$ is an object A so that

$$\dim \mathrm{Hom}(A, A[i]) = \begin{cases} 1 & i = 0, 2 \\ 0 & i \neq 0, 2 \end{cases}.$$

Let $\mathcal{C}$ be a Calabi-Yau category. Given a spherical object A and another object B, Seidel and Thomas define the **spherical twist** $\sigma_A(B)$, defined up to isomorphism by the existence of an exact triangle

$$R\mathrm{Hom}^\bullet(A, B) \otimes_k A \to B \to \sigma_A(B) \to .$$

This defines σ_A as a function from isomorphism classes of objects to isomorphism classes of objects. In most scenarios, σ_A can be upgraded to a functor $\mathcal{C} \to \mathcal{C}$. However, we shall not need the functoriality here.

4.2.2 Examples of Calabi-Yau categories

Modules for Π_Q

We define the categories $\mathbf{rep}_{fd}(\Pi), \mathbf{rep}_n(\Pi), \mathsf{D}^b_{fd}(\Pi)$, and $\mathsf{D}^b_n(\Pi)$ to be, respectively, the category of finite-dimensional modules for Π, the category of nilpotent modules for Π, the bounded derived category of modules for Π with finite-dimensional cohomology, and the bounded derived category of modules for Π with nilpotent cohomology.

Theorem 4.2.3. *If Q is a connected quiver with no loops, and Q is not a Dynkin quiver, then $\mathsf{D}^b_{fd}(\Pi)$ and $\mathsf{D}^b_n(\Pi)$ are 2-Calabi-Yau categories.*

It turns out that when Q is a Dynkin quiver, then $\mathsf{D}^b_{fd}(\Pi)$ and $\mathsf{D}^b_n(\Pi)$ are not 2-Calabi-Yau. In this case, we let Q' be an extended Dynkin quiver containing Q and define $\mathsf{D}^b_n(\widetilde{\Pi})$ to be the triangulated subcategory of $\mathsf{D}^b_n(\Pi_{Q'})$ generated by objects $\{S_i\}_{i\in Q_0}$; then $\mathsf{D}^b_n(\widetilde{\Pi})$ is Calabi-Yau. Similarly we can define $\mathsf{D}^b_n(\widetilde{\Pi})$ for a general quiver by performing this replacement for each component of Q that is Dynkin.

When $\mathsf{D}^b_{fd}(\Pi)$ is Calabi-Yau, then the simple module S_i is a spherical object. The spherical twist $\sigma_i(B) \coloneqq \sigma_{S_i}(B)$ coincides with the **reflection functor** of Baumann and Kamnitzer [9]. We will discuss it further in Section 4.3.

A motivating philosophy for considering preprojective algebras asserts that the categories $\mathsf{D}^b_n(\widetilde{\Pi})$ are universal examples of Calabi-Yau categories generated by spherical objects. To be more precise, let D be a Calabi-Yau category and $\mathsf{D}^\heartsuit$ the heart of a *t-structure* on D (see Section 4.4.1). If $\{S_i\}_{i\in I}$ is a collection of spherical objects in $\mathsf{D}^\heartsuit$ so that $\mathrm{Hom}(S_i, S_j) = 0$ when $i \neq j$, then the triangulated subcategory of D generated by $\{S_i\}_{i\in I}$ is equivalent to $\mathsf{D}^b_n(\widetilde{\Pi}_Q)$. Here Q is a quiver with vertex set I and so that the double quiver $\overline{Q}$ has $\dim \mathrm{Ext}^1(S_i, S_j)$ many arrows from i to j. Furthermore, the extension-closed abelian subcategory of $\mathsf{D}^\heartsuit$ generated by $\{S_i\}_{i\in I}$ is equivalent to $\mathbf{rep}_n(\Pi_Q)$. Hence the main results of this chapter, although stated in terms of $\mathbf{rep}_n(\Pi_Q)$, in fact extend to many other Calabi-Yau categories. In the remainder of this section we shall see some motivating examples.

Coherent sheaves

Let X be a smooth two-dimensional variety over k with trivial canonical bundle. For instance, X could be a K3 surface or a smooth affine surface. Let $\mathsf{D}_{coh,cmp}(X)$ be the bounded derived category of sheaves of $\mathcal{O}_X$-modules on X with coherent and compactly supported cohomology. Then Grothendieck coherent duality implies that $\mathsf{D}_{coh,cmp}(X)$ is a Calabi-Yau category. For example, this applies when X is the affine space $\mathbb{A}^2$. If X is projective, then $\mathsf{D}_{coh,cmp}(X) = \mathsf{D}_{coh}(X)$ is the usual derived category of coherent sheaves. In particular, when X is an abelian surface (such as the product of two elliptic curves) or a K3 surface, then $\mathsf{D}_{coh}(X)$ is a Calabi-Yau category.

McKay correspondence

Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. The **McKay correspondence** associates an affine-type quiver Q_G to G. One characterization of this quiver is the following:

Theorem 4.2.4. *Let $G \subseteq \mathrm{SL}_2(\mathbb{C})$ be finite. Then the category $\mathbf{QCoh}_G(\mathbb{A}^2)$, consisting of G-equivariant quasi-coherent sheaves on $\mathbb{A}^2$, is equivalent to the category of Π_{Q_G}-modules. Here, Q_G is the quiver associated to G by the McKay correspondence.*

We note that since $\mathbb{A}^2$ is an affine variety, the category $\mathbf{QCoh}_G(\mathbb{A}^2)$ can be explicitly modeled as the category of modules for the semidirect product algebra $k[x_1, x_2] \rtimes k[G]$. As a consequence of the theorem, the category of G-equivariant quasi-coherent sheaves with a finite-dimensional space of global sections is equivalent to $\mathbf{rep}_{fd}(\Pi_{Q_G})$. Hence the category $\mathsf{D}_{coh,cmp,G}(\mathbb{A}^2)$ is equivalent to $\mathsf{D}^b_{fd}(\Pi_{Q_G})$. In particular, it is 2-Calabi-Yau. There is a geometric interpretation for the fact that $\mathsf{D}_{coh,cmp,G}(\mathbb{A}^2)$ is a 2-Calabi-Yau category, which we now explain.

Set X to be the orbit space $\mathbb{C}^2/G$. Then X has the structure of an affine variety: letting $R = k[x_1, x_2]$, viewed as the ring of polynomial functions on $\mathbb{C}^2$, there is an induced action of G on R. The ring of invariants R^G consists of the functions which are well-defined on X, and we declare X to be the affine variety $\operatorname{Spec} R^G$. If G is non-trivial, then X is a singular variety (an *ADE singularity*). Any surface singularity has a *minimal resolution*, which is a proper birational map $Y \to X$ with Y smooth, so that any other such map factors through it.

Theorem 4.2.5 ([16]). *Let X be an ADE singularity, and let $Y \to X$ be the minimal resolution of X. Then there is an equivalence of categories $\mathbf{Coh}(Y) \xrightarrow{\sim} \mathbf{Coh}_G(\mathbb{A}^2)$.*

Furthermore, if $Y \to X$ is the minimal resolution, then the canonical bundle ω_Y is the trivial bundle $\mathcal{O}_Y$ (the resolution is *crepant*). As a result, the previous subsection implies that $\mathsf{D}_{coh,cmp}(Y) \cong \mathsf{D}_{coh,cmp,G}(\mathbb{A}^2)$ is Calabi-Yau.

4.3 Bricks and reflection functors

Fix a finite quiver Q with no loops (arrows a with $s(a) = t(a)$) and let Π_Q be its preprojective algebra. Then Q defines a symmetric Cartan matrix $(A_{ij})_{i,j \in Q_0}$, where for $i \neq j$, the value of A_{ij} is the number of arrows with source and target both in the set $\{i, j\}$. Let Δ_Q denote the simplicial Kac–Moody root system of A_{ij} (as in Section 2.3.4). Let $\mathbb{Z}\Delta_Q$ be the $\mathbb{Z}$-lattice spanned by Δ_Q (the *root lattice*). Taking dimension vectors induces a map $\underline{\dim} : K_0(\mathbf{rep}_{fd}(\Pi_Q)) \to \mathbb{Z}\Delta$ from the Grothendieck group of $\mathbf{rep}_{fd}(\Pi_Q)$ to the root lattice which restricts to an isomorphism $K_0(\mathbf{rep}_n(\Pi_Q)) \xrightarrow{\sim} \mathbb{Z}\Delta$. The bilinear form on $\mathbb{Z}\Delta$ induced by the Cartan matrix satisfies the identity

$$(\underline{\dim}\, M, \underline{\dim}\, N) = \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}^1(M, N) + \dim \operatorname{Hom}(N, M).$$

When Q is connected and non-Dynkin, the right hand side coincides with the *Euler form* on $K_0(\mathbf{rep}_n(\Pi_Q))$, which is defined by $([M], [N]) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \operatorname{Hom}(M, N[i])$.

Let W be the Weyl group of Δ. Then the action of $s_i \in W$ on $\mathbb{Z}\Delta$ is categorified by a functor $\mathbf{R}\sigma_i : \mathsf{D}^b_{fd}(\Pi_Q) \to \mathsf{D}^b_{fd}(\Pi_Q)$, given by the spherical twist around the simple module S_i [16]. We shall focus on its underived version, the *reflection functors* of Baumann–Kamnitzer [9] (which generalize the Bernstein–Gelfand–Ponomarev reflection functors for representations of Q).

We use the presentation given in [22]. Write $\mathbf{NoSub}_i$ for the subcategory of $\mathbf{rep}_{fd}(\Pi_Q)$ consisting of modules that do not have S_i as a submodule, and $\mathbf{NoQuot}_i$ for the subcategory of $\mathbf{rep}_{fd}(\Pi_Q)$ consisting of modules that do not have S_i as a quotient.

Let M be a Π_Q-module. We write $\overline{Q}_1$ for the arrows of the double quiver of Q, and for each $a \in Q_1$ we set $\varepsilon(a) = 1$ and $\varepsilon(a^*) = -1$. Define the vector space $M_{\to i} = \bigoplus_{\substack{a \in Q_1 \\ t(a)=i}} M_{s(a)}$ and maps $M_{\mathrm{in}(i)} = (\varepsilon(a)L_a)_{\substack{a \in Q_1 \\ t(a)=i}}$ and $M_{\mathrm{out}(i)} = (L_a)_{\substack{a \in Q_1 \\ s(a)=i}}$. We then have the diagram

$$M_{\to i} \xrightarrow{M_{\mathrm{in}(i)}} M_i \xrightarrow{M_{\mathrm{out}(i)}} M_{\to i}. \tag{4.3.1}$$

By the preprojective algebra relation, $M_{\mathrm{in}(i)} M_{\mathrm{out}(i)} = 0$.

The reflection functor σ_i replaces (4.3.1) with

$$M_{\to i} \xrightarrow{M_{\mathrm{out}(i)} M_{\mathrm{in}(i)}} \ker(M_{\mathrm{in}(i)}) \hookrightarrow M_{\to i}.$$

The reflection functor σ_i^{-1} replaces (4.3.1) with

$$M_{\to i} \twoheadrightarrow \operatorname{coker}(M_{\mathrm{out}(i)}) \xrightarrow{M_{\mathrm{out}(i)} M_{\mathrm{in}(i)}} M_{\to i}.$$

We have that

$$\mathrm{Hom}(M, \sigma_i N) = \mathrm{Hom}(\sigma_i^{-1} M, N),$$

so in particular σ_i preserves limits and σ_i^{-1} preserves colimits.

Proposition 4.3.1. *Reflection functors give inverse equivalences of categories*

$$\sigma_i : \mathbf{NoQuot}_i \rightleftarrows \mathbf{NoSub}_i : \sigma_i^{-1}.$$

Furthermore, if $M \in \mathbf{NoQuot}_i$ then $\underline{\dim}\, \sigma_i M = s_i \,\underline{\dim}\, M$, and if $M \in \mathbf{NoSub}_i$ then $\underline{\dim}\, \sigma_i^{-1} M = s_i \,\underline{\dim}\, M$.

We could extend the domain of σ_i to $\mathbf{rep}_{fd}(\Pi_Q)$ by taking the same definition, but we use the convention here to make some statements easier. In particular, whenever σ_i is applied to a module M, we assume that M is in $\mathbf{NoQuot}_i$, and similarly for σ_i^{-1} and $\mathbf{NoSub}_i$.

Definition 4.3.2. A module M is called a **brick** if $\mathrm{Hom}_{\Pi_Q}(M,M) = k$ (equivalently, if every nonzero map $M \to M$ is invertible). A brick M is called **real** if $\underline{\dim}(M)$ is in Δ^+_{re}, and **imaginary** if $\underline{\dim}(M)$ is in Δ^+_{im}. We write $\mathsf{Bricks}(\Pi_Q)$ for the set of isomorphism classes of bricks of Π_Q.

By Schur's lemma, in any abelian subcategory of $\mathbf{rep}_{fd}(\Pi_Q)$ the simple objects will be bricks. Bricks have been studied in the context of stability since, for instance, the moduli spaces of bricks are better behaved than the moduli of more general modules. They are also important in the study of torsion classes, as we will see.

The following lemma is a strengthening of [22, Proposition 4.14]. The fact that the dimension vectors of bricks for Π_Q are always roots seems to not appear in the literature, although the analogous statement is well-known for bricks of $k[Q]$ and the proof is similar.

Lemma 4.3.3. *Every brick is either real or imaginary. If Q is a connected non-Dynkin quiver, then a module M is a real brick if and only if*

$$\mathrm{Ext}^i(M,M) = \begin{cases} k & \text{if } i = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}.$$

If Q is an affine quiver, then a module M is an imaginary brick if and only if

$$\mathrm{Ext}^i(M,M) = \begin{cases} k & \text{if } i = 0 \text{ or } 2 \\ k^{\oplus 2} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let B be a brick with $\underline{\dim}\, B = \sum_i c_i \alpha_i$. If there is some i so that $\langle \alpha_i^\vee, \underline{\dim}\, B\rangle > 0$, then by [22, Proposition 4.11] we have that B is in either $\mathbf{NoQuot}_i$ or $\mathbf{NoSub}_i$. Then

$B = \sigma_i^{-1}B'$ or $B = \sigma_i B'$, respectively, for some brick B' with dimension vector $\underline{\dim}\, B' = s_i\, \underline{\dim}\, B = \underline{\dim}\, B - \langle \alpha_i^\vee, \underline{\dim}\, B\rangle$. By induction on height, B' is either real or imaginary. Since Δ_Q is preserved under the action of W, B is also real or imaginary.

Otherwise, $\langle \alpha_i^\vee, \underline{\dim}\, B\rangle \leq 0$ for all i. Furthermore, the support of $\underline{\dim}\, B$ is connected in the sense of Proposition 2.3.13, since modules with disconnected support are decomposable and bricks are indecomposable. Hence by Proposition 2.3.13, the dimension $\underline{\dim}\, B$ is an imaginary root.

To see the statement about Ext groups, note that $\mathsf{D}^b_{fd}(\Pi_Q)$ is 2-Calabi-Yau, so it is enough to compute $\mathrm{Ext}^0(M, M) = \mathrm{Hom}(M, M)$ and $\mathrm{Ext}^1(M, M)$. The former is k if and only if M is a brick. The latter is 0 if and only if M is a real brick by [22, Proposition 4.14]. Hence we just need to check that if Q is an affine quiver and M is an imaginary brick, then $\mathrm{Ext}^1(M, M) = k^{\oplus 2}$. This follows from the Euler formula, since if $\underline{\dim}(M) = r\delta$, then

$$(r\delta, r\delta) = 0 = \dim \mathrm{Hom}(M, M) - \dim \mathrm{Ext}^1(M, M) + \dim \mathrm{Ext}^2(M, M) = 2 - \dim \mathrm{Ext}^1(M, M).$$

□

Remark 4.3.4. As a consequence of the lemma, for an affine quiver Q the spherical objects of $\mathbf{rep}_{fd}(\Pi_Q) \subseteq \mathsf{D}^b_{fd}(\Pi_Q)$ coincide with the real bricks of Π_Q. Furthermore, the imaginary bricks correspond to objects with the cohomology groups of a torus. In the context of mirror symmetry, imaginary bricks should correspond to special Lagrangian tori in a (non-compact) Calabi-Yau 2-fold X, and the SYZ conjecture predicts that the mirror variety to X can be built from the moduli space of such tori. In support of this prediction, it turns out that the minimal resolution of the ADE singularity associated to Q discussed in Section 4.2.2 can be realized as the moduli space of certain imaginary bricks for Π_Q [39].

4.4 Torsion classes

4.4.1 Torsion classes and t-structures

In this section and what follows, we assume any abelian or triangulated categories that appear are essentially small.

Definition 4.4.1. A **torsion class** in an abelian category A is a collection of objects $\mathcal{T}$ satisfying the following properties:

- if M is in $\mathcal{T}$ and M' is isomorphic to M then M' is in $\mathcal{T}$, and
- if M is in $\mathcal{T}$ and Q is a quotient object of M then Q is in $\mathcal{T}$, and
- if A is a subobject of M and A and M/A are both in $\mathcal{T}$ then M is in $\mathcal{T}$.

We denote the poset of torsion classes under inclusion order by $\mathsf{Tors}(\mathsf{A})$.

The collection of torsion classes is closed under intersection, so any set of modules $\mathcal{S}$ is contained in a unique minimal torsion class $\overline{\mathcal{S}}$. In particular, $\mathsf{Tors}(\Pi_Q)$ is a complete lattice. Torsion classes are determined by the bricks that they contain, in the sense that if $\mathcal{T}$ is a torsion class, then $\mathcal{T} = \overline{\mathcal{T} \cap \mathsf{Bricks}(\mathsf{A})}$ [24]. A **torsion-free class** is defined dually to torsion classes, so that it is closed under extensions and subobjects. Given any set $\mathcal{X}$ of objects of A, define

$$\mathcal{X}^{\perp} = \{M \in \mathsf{A} \mid \forall X \in \mathcal{X},\ \mathrm{Hom}(X, M) = 0\}.$$

Then $\mathcal{X}^{\perp}$ is always a torsion-free class. Similarly, if $\mathcal{Y}$ is any set of objects of A, then

$${}^{\perp}\mathcal{Y} \coloneqq \{M \in \mathsf{A} \mid \forall Y \in \mathcal{Y},\ \mathrm{Hom}(M, Y) = 0\}$$

is a torsion class. A **torsion pair** is a pair $(\mathcal{T}, \mathcal{F})$ so that $\mathcal{T}^{\perp} = \mathcal{F}$ and ${}^{\perp}\mathcal{F} = \mathcal{T}$. Torsion classes biject with torsion pairs, since for any set of objects $\mathcal{X}$ the collection $\overline{\mathcal{X}} \coloneqq {}^{\perp}(\mathcal{X}^{\perp})$ is the minimal torsion class containing $\mathcal{X}$.

If two submodules $M_1, M_2 \subseteq M$ are in a torsion class $\mathcal{T}$, then the submodule $M_1 + M_2$ that they generate is also in $\mathcal{T}$. Hence there is a unique maximal submodule of M in $\mathcal{T}$.

Definition 4.4.2. Given an abelian category A and an object M of A, let $\mathcal{T}(M)$ denote the minimal torsion class containing M. If $\mathcal{T}$ is any torsion class, let $\Xi_{\mathcal{T}}(M)$ denote the unique maximal subobject of M in $\mathcal{T}$.

Torsion classes on an abelian category are closely related to another structure on its derived category.

Definition 4.4.3. Let D be a triangulated category. A ***t*-structure** on D consists of full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ such that:

- $\mathsf{D}^{\leq 0}[1] \subseteq \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}[-1] \subseteq \mathsf{D}^{\geq 0}$, and
- $\mathrm{Hom}(M, N[-1]) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 0}$, and
- For each $M \in \mathsf{D}$ there is an exact triangle $\tau_{\leq 0}M \to M \to \tau_{\geq 1}M \to$ so that $\tau_{\leq 0} \in \mathsf{D}^{\leq 0}$ and $\tau_{\geq 1}M \in \mathsf{D}^{\geq 0}[-1]$.

A t-structure is determined by the subcategory $\mathsf{D}^{\leq 0}$. The **heart** of a t-structure is the abelian category $\mathsf{D}^{\heartsuit} := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. If A is an abelian category, then the **standard *t*-structure** on $\mathsf{D}^b(\mathsf{A})$ has $\mathsf{D}^{b^{\leq 0}}(\mathsf{A}) := \{M \mid \forall i > 0,\ \mathsf{H}^i(M) = 0\}$ and $\mathsf{D}^{b^{\geq 0}}(\mathsf{A}) := \{M \mid \forall i < 0,\ \mathsf{H}^i(M) = 0\}$. Given a t-structure $\mathsf{D}^{\leq 0}$, the maps $M \mapsto \tau_{\leq 0}M$ and $\tau_{\geq 1}M$ extend to **truncation functors** $\tau_{\leq 0} : \mathsf{D} \to \mathsf{D}^{\leq 0}$ and $\tau_{\geq 0} : \mathsf{D} \to \mathsf{D}^{\geq 0}$. The **cohomology functor** of the t-structure is $\mathsf{H}^i_t : \mathsf{D} \to \mathsf{D}^{\heartsuit}$, given by $M \mapsto \tau_{\leq i}\tau_{\geq i}M$.

Each torsion pair $(\mathcal{T}, \mathcal{F})$ for an abelian category A gives rise to a t-structure on its derived category $\mathsf{D}^b(\mathsf{A})$. More generally, let D be a triangulated category with t-structure $\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0}$. Then for a torsion pair $(\mathcal{T}, \mathcal{F})$ in the heart $\mathsf{D}^{\heartsuit}$, the Happel–Reiten–Smalø**tilt** of the t-structure by $(\mathcal{T}, \mathcal{F})$ is the t-structure

$$\mathsf{D}'^{\leq 0} := \{X \in \mathsf{D} \mid \mathsf{H}^i_t(X) = 0 \text{ if } i > 0 \text{ and } \mathsf{H}^0(X) \in \mathcal{T}\}$$

$$\mathsf{D}'^{\geq 0} := \{X \in \mathsf{D} \mid \mathsf{H}^i_t(X) = 0 \text{ if } i < -1 \text{ and } \mathsf{H}^{-1}(X) \in \mathcal{F}\}.$$

The truncation functor $\tau'_{\leq 0}$ for the tilt is a generalization of $\Xi_{\mathcal{T}}$: if $M \in \mathsf{D}^{\heartsuit}$, then $\tau'_{\leq 0}M = \Xi_{\mathcal{T}}(M)$.

The tilts of a given t-structure $\mathsf{D}^{\leq 0}$ are exactly the **intermediate t-structures**, which are t-structures $\mathsf{D}'^{\leq 0}$ such that $\mathsf{D}^{\leq -1} \subseteq \mathsf{D}'^{\leq 0} \subseteq \mathsf{D}^{\leq 0}$.

4.4.2 Lattice structure, wide subcategories, and brick labelings

Let $\mathsf{Tors}(\mathsf{A})$ denote the collection of torsion classes in A, ordered by inclusion. We also write $\mathsf{Tors}(\Pi_Q)$ to mean $\mathsf{Tors}(\mathbf{rep}_n(\Pi_Q))$. Let $\mathsf{Bricks}(\mathsf{A})$ be the set of (isomorphism classes of) bricks in A. Define relations $\to, \twoheadrightarrow, \hookrightarrow$ on $\mathsf{Bricks}(\mathsf{A})$ as follows. There is an arrow $B_1 \to B_2$ if $\mathrm{Hom}(B_1, B_2) \neq 0$, an arrow $B_1 \hookrightarrow B_2$ if B_1 has a filtration with successive subquotients that are submodules of B_2, and an arrow $B_1 \twoheadrightarrow B_2$ if B_2 has a filtration with successive subquotients that are quotients of B_1. (See also Example 2.4.23.) The following is essentially shown in [46].

Proposition 4.4.4. *If A is a finite-length abelian category, then $\mathsf{Tors}(\mathsf{A})$ is a completely semidistributive lattice. Furthermore, $(\mathsf{Bricks}(\mathsf{A}), \to, \twoheadrightarrow, \hookrightarrow)$ is a two-acyclic factorization system so that every arrow has an image and co-image, and $\mathsf{Tors}(\mathsf{A}) \cong \mathrm{Pairs}(\mathsf{Bricks}(\mathsf{A}), \to)$.*

As a result, by Proposition 2.4.27, each forcing-open subset of $\mathsf{Bricks}(\mathsf{A})$ determines a lattice quotient of $\mathsf{Tors}(\mathsf{A})$. Hence we are interested in the forcing relation on bricks.

Lemma 4.4.5. *Let B_1 and B_2 be bricks in A. If B_1 directly forces B_2, then either B_1 is a submodule of B_2 or B_1 is a quotient of B_2. In particular, if B_1 and B_2 are non-isomorphic bricks in $\mathbf{rep}_n(\Pi_Q)$ so that B_1 forces B_2 in $\mathsf{Tors}(\Pi_Q)$, then $\underline{\dim}\, B_1 < \underline{\dim}\, B_2$ in the root poset.*

Proof. If B_1 directly forces B_2, then either $B_1 \hookrightarrow B_2$ and there is no brick $B \neq B_1$ so that $B_1 \twoheadrightarrow B \hookrightarrow B_2$, or else $B_2 \twoheadrightarrow B_1$ and there is no brick $B \neq B_1$ so that $B_2 \twoheadrightarrow B \hookrightarrow B_1$. We consider the first case, the other is similar. In the first case, B_1 has a filtration $0 \subsetneq$

$B_1^1 \subsetneq \cdots \subsetneq B_1^k = B_1$ so that successive subquotients are submodules of B_2. If $k > 1$, then B_1^k/B_1^{k-1} is a proper quotient of B_1. There is some brick quotient B of B_1^k/B_1^{k-1} which is also a subobject of B_1^k/B_1^{k-1} (for instance, by repeatedly taking images of non-invertible endomorphisms). Then $B_1 \twoheadrightarrow B$, since B is a quotient of B_1, and $B \hookrightarrow B_2$, since B is a submodule of B_2. This is a contradiction, so we must have $k = 1$. In that case, $B = B_1^1/B_1^0$ is a submodule of B_2, so we are done. □

Lemma 4.4.6. *Assume $k = \mathbb{F}_p$. Let A be a k-algebra which is a quotient of $k[Q]$ for some finite quiver Q. Then the lattice $\mathsf{Tors}(\mathbf{rep}_{fd}(A))$ is profinite.*

Proof. By Lemma 4.4.5, the set of bricks with dimension $> d$ is forcing-closed. Contracting these bricks gives a complete lattice quotient of $\mathsf{Tors}(\mathbf{rep}_{fd}(A))$ so that the uncontracted bricks have dimension at most d. There are finitely many $k[Q]$-algebra structures on a vector space of dimension at most d, so there are finitely many isomorphism classes of bricks. Hence the lattice quotient is a finite lattice. We may distinguish any two elements of $\mathsf{Tors}(\mathbf{rep}_{fd}(A))$ by taking d large enough, so $\mathsf{Tors}(\mathbf{rep}_{fd}(A))$ is profinite. □

The cover relations in $\mathsf{Tors}(\mathsf{A})$ are particularly well-behaved. The following summarizes results of the appendix to [38].

Proposition 4.4.7. *If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 \lessdot \mathcal{T}_2$ is a cover relation if and only if there is a unique brick in $\mathcal{T}_1^{\perp} \cap \mathcal{T}_2$. Furthermore, a maximal chain $\mathcal{C}$ in $\mathsf{Tors}(\mathsf{A})$ is determined uniquely by the sequence of bricks labeling its cover relations. Any sequence of bricks $(B_i)_{i \in I}$ so that $i < j$ implies $\mathrm{Hom}(B_i, B_j) = 0$ appears as a subsequence of the cover relations on some maximal chain in $\mathsf{Tors}(\mathsf{A})$.*

The brick associated to a cover relation $\mathcal{T}_1 \lessdot \mathcal{T}_2$ is called the **brick label** of the cover.

Definition 4.4.8. Let A be an abelian category. A **wide subcategory** of A is a subcategory W so that:

- if M is in W and M' is isomorphic to M then M' is in W, and

- if M, N are in W and $f : M \to N$ is a map, then the kernel and cokernel of f are in W, and
- if A is a subobject of M and A and M/A are both in W then M is in W.

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ are torsion classes so that $\mathcal{T}_1^{\perp} \cap \mathcal{T}_2$ is wide, then we say the interval $[\mathcal{T}_1, \mathcal{T}_2]$ is a **wide interval**.

We write $\mathsf{Wide}(X)$ for the wide subcategory generated by a set of objects X. We summarize results from [2].

Proposition 4.4.9. *An interval $[\mathcal{T}_1, \mathcal{T}_2]$ is wide if and only if $\mathcal{T}_2$ is the join of $\{\mathcal{T} \in \mathsf{Tors}(\mathsf{A}) \mid \mathcal{T}_1 \lessdot \mathcal{T} \leq \mathcal{T}_2\}$, which occurs if and only if $\mathcal{T}_1$ is the meet of $\{\mathcal{T} \in \mathsf{Tors}(\mathsf{A}) \mid \mathcal{T}_1 \leq \mathcal{T} \lessdot \mathcal{T}_2\}$. In this case, there is a brick label-preserving lattice isomorphism $[\mathcal{T}_1, \mathcal{T}_2] \xrightarrow{\sim} \mathsf{Tors}(\mathcal{T}_1^{\perp} \cap \mathcal{T}_2)$. In particular, the brick labels of the upper covers of $\mathcal{T}_1$ in $[\mathcal{T}_1, \mathcal{T}_2]$ coincide with the brick labels of the lower covers of $\mathcal{T}_2$, which coincide with the simple objects of $\mathcal{T}_1^{\perp} \cap \mathcal{T}_2$.*

4.4.3 Stability conditions and Harder–Narasimhan filtrations

A common source of torsion pairs comes from *stability conditions*. We first will consider numerical stability conditions in the sense of King. Let Q be a quiver with vertex set Q_0, and let V be the vector space with basis $\{\alpha_v\}_{v \in Q_0}$. Then a **stability condition** on $\mathbf{rep}_{fd}(\Pi_Q)$ is an element $\theta \in V^*$. In this case, if $M \in \mathbf{rep}_{fd}(\Pi_Q)$ then we write $\theta(M)$ or $\langle \theta, M \rangle$ for the pairing $\langle \theta, \underline{\dim}\, M \rangle$. Given a stability condition, there are two associated torsion classes:

$$\mathcal{T}(\theta) := \{M \in \mathbf{rep}_{fd}(\Pi_Q) \mid \forall N \subseteq M,\ \theta(N) \leq \theta(M)\}$$

$$\mathcal{T}_*(\theta) := \{M \in \mathbf{rep}_{fd}(\Pi_Q) \mid \forall N \subsetneq M,\ \theta(N) < \theta(M)\}.$$

Then $\mathcal{T}_*(\theta) \subseteq \mathcal{T}(\theta)$ and the interval $[\mathcal{T}_*(\theta), \mathcal{T}(\theta)]$ is wide. We write $\mathsf{Wide}(\theta) := \mathcal{T}_*(\theta)^{\perp} \cap \mathcal{T}(\theta)$. Objects of $\mathsf{Wide}(\theta)$ are called **θ-semistable**, and the simple objects are called **θ-stable**. An object M is θ-semistable (resp. θ-stable) if and only if $\theta(M) = 0$ and for all $N \subsetneq M$, $\theta(N) \leq 0$ (resp. $\theta(N) < 0$). We will write $\Xi_\theta(M)$ to mean $\Xi_{\mathcal{T}(\theta)}(M)$.

Given a module M, we write $\mathsf{Stab}(M) = \{\theta \in V^* \mid M \text{ is } \theta\text{-stable}\}$. For most modules, $\mathsf{Stab}(M)$ is empty. If $\mathsf{Stab}(M) \neq \varnothing$ then M is a brick, by Schur's lemma. But there can still be bricks with $\mathsf{Stab}(M)$ empty [22, Theorem 7.2].

Definition 4.4.10. A **shard module** is a real brick M with $\mathsf{Stab}(M) \neq \varnothing$.

The following was conjectured in [22, Conjecture 6.11]. We shall prove it in Corollary 4.5.8.

Conjecture 4.4.11. *Let M_1, M_2 be real shard modules for Π_Q so that $\underline{\dim}\, M_1 = \underline{\dim}\, M_2$. Then* $\mathrm{Hom}(M_1, M_2) \neq 0$.

Lemma 4.4.12. *Let $\theta \in V^*$ be a stability condition so that there is a unique positive real root $\alpha \in \Delta^+_{\mathrm{re}}$ with $\langle \theta, \alpha \rangle = 0$. Then there is a unique θ-stable real brick.*

Proof. If M and M' are distinct simple objects of $\mathsf{Wide}(\theta)$, then $\mathrm{Hom}(M', M) = \mathrm{Hom}(M, M') = 0$. If M and M' are both real bricks, then $\underline{\dim}\, M = \underline{\dim}\, M' = \alpha$ since $\langle \theta, \underline{\dim}\, M \rangle = \langle \theta, \underline{\dim}\, M' \rangle = 0$. It would follow that

$$\begin{aligned} 0 < (\underline{\dim}\, M', \underline{\dim}\, M) &= \dim \mathrm{Hom}(M', M) - \dim \mathrm{Ext}^1(M', M) + \dim \mathrm{Hom}(M, M') \\ &= -\dim \mathrm{Ext}^1(M', M) \end{aligned}$$

which is impossible. Hence there is at most one θ-stable real brick. Conversely, [46, Theorem 5.7] implies that

$$\bigcup_{\substack{M \in \mathsf{Bricks}(\Pi_Q) \\ \underline{\dim}\, M = \alpha}} \mathsf{Stab}(M) = \alpha^\perp \setminus \{\beta_1^\perp, \ldots, \beta_k^\perp\}$$

for some finite set of real roots $\beta_1, \ldots, \beta_k$. In particular, if θ is any element of $\alpha^\perp \setminus \{\beta^\perp \mid \beta \in \Delta^+_{\mathrm{re}}\}$, then there exists a θ-stable module of dimension vector α. $\square$

Definition 4.4.13. Let M be a Π_Q-module. The **Harder–Narasimhan polytope** $\mathsf{HN}(M)$ is the convex hull of $\{\underline{\dim}\, N \mid N \subseteq M\}$ in V.

A real brick M is a shard module if and only if there is an edge of $\mathsf{HN}(M)$ connecting 0 to $\underline{\dim}(M)$. The following is an analog of one of the main results in *Harder–Narasimhan theory*; note that in our setting of finite-length abelian categories it follows directly from properties of torsion classes and bricks. The filtration of M constructed in the theorem is called a **Harder–Narasimhan filtration**.

Theorem 4.4.14. *Let $f, g \in V^*$ be such that $g(\Pi) > 0$ and the function $\alpha \mapsto f(\alpha)/g(\alpha)$ is injective on Δ^+. Then for any module M, the function $\mu \mapsto \Xi_{f-g\mu}(M)$ from real numbers to submodules of M is an increasing piecewise-constant function with finitely many discontinuities. If N is a value of the function, then $\underline{\dim}\, N$ is a vertex of $\mathsf{HN}(M)$. If N^- and N^+ are the left and right limits of a discontinuity at μ, then $\underline{\dim}\, N^-$ and $\underline{\dim}\, N^+$ are connected by an edge in $\mathsf{HN}(M)$ and N^+/N^- is in $\mathsf{Wide}(f - g\mu)$.*

We shall see pairs f, g as in the theorem statement quite frequently; for instance, they will also be the source of *reflection orders* in Chapter 5. Such pairs give rise to *Bridgeland stability conditions*; the idea is to combine f and g into a complex valued function $f + ig$. We summarize some of the theory as applied to our setting. See [17] for more details. (Note that there the function Z is allowed to take purely real negative values; for simplicity, we disallow this.)

Definition 4.4.15. Let A be a finite-length abelian category. A **Bridgeland stability function** on A is a group homomorphism $Z : K_0(\mathsf{A}) \to \mathbb{C}$ so that for every nonzero object $M \in \mathsf{A}$, the value of $Z(M)$ has positive imaginary part. The **phase** of a nonzero object M, denoted $\phi(M)$, is the unique $\phi \in (0, 1)$ so that $Z(M) = r \exp(i\phi\pi)$ for some $r > 0$.

For each $\phi \in [0, 1]$, we define torsion classes

$$\mathcal{T}(\geq \phi) \coloneqq \{M \in \mathsf{A} \mid \forall N \subsetneq M,\ \phi(M/N) \geq \phi\}$$

$$\mathcal{T}(> \phi) \coloneqq \{M \in \mathsf{A} \mid \forall N \subsetneq M,\ \phi(M/N) > \phi\}$$

and the wide subcategory

$$\mathsf{Wide}(=\phi) = \mathcal{T}(>\phi)^{\perp} \cap \mathcal{T}(\geq \phi) = \{M \in \mathsf{A} \mid \phi(M) = \phi \text{ and } \forall N \subseteq M,\ \phi(M/N) \geq \phi\}.$$

Then $\mathcal{C}_Z := \{\mathcal{T}(\geq \phi) \mid \phi \in [0,1]\} \cup \{\mathcal{T}(>\phi) \mid \phi \in [0,1]\}$ is a chain in $\mathsf{Tors}(\mathsf{A})$. We say that a chain $\mathcal{C}$ in $\mathsf{Tors}(\mathsf{A})$ is **wide** if it is closed under unions and intersections and for every pair of torsion classes $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ which form a cover relation in $\mathcal{C}$, the interval $[\mathcal{T}_1, \mathcal{T}_2]_{\mathsf{Tors}(\mathsf{A})}$ is wide.

Lemma 4.4.16. *If Z is a Bridgeland stability function on A, then $\mathcal{C}_Z$ is a wide chain. Furthermore, every cover relation in $\mathcal{C}_Z$ is of the form $\mathcal{T}(>\phi) \subsetneq \mathcal{T}(\geq \phi)$ for some phase ϕ.*

Proof. First, $\mathcal{C}_Z$ is closed under unions: if $\{\mathcal{T}_i\}_{i\in I}$ is a family of torsion classes in $\mathcal{C}_Z$ associated to phases $\{\phi_i\}_{i\in I}$, then set $\phi := \inf_{i\in I}\phi_i$. If $\mathcal{T}(\geq \phi)$ is among the $\mathcal{T}_i$, then $\bigcup_{i\in I}\mathcal{T}_i = \mathcal{T}(\geq \phi) \in \mathcal{C}_Z$. Otherwise, $\bigcup_{i\in I}\mathcal{T}_i = \mathcal{T}(>\phi) \in \mathcal{C}_Z$. The proof that $\mathcal{C}_Z$ is closed under intersection is similar.

Now let $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ be a cover relation in $\mathcal{C}_Z$. Let $\phi_1 = \sup\{\phi \mid \mathcal{T}(>\phi) \subseteq \mathcal{T}_1\}$ and $\phi_2 = \inf\{\phi \mid \mathcal{T}_2 \subseteq \mathcal{T}(\geq \phi)\}$. We claim that $\phi_1 = \phi_2$, so that $\mathcal{T}_1 = \mathcal{T}(>\phi_1)$ and $\mathcal{T}_2 = \mathcal{T}(\geq \phi_1)$. If not, then $\mathcal{T}_1 = \mathcal{T}(\geq \phi_1)$ and $\mathcal{T}_2 = \mathcal{T}(>\phi_2)$, so there is a module M in $\mathcal{T}(\geq \phi_1)^{\perp} \cap \mathcal{T}(>\phi_2)$. If ϕ_1' and ϕ_2' denote the minimum and maximum phases of the semistable composition factors of M, then $\phi_1 > \phi_1' \geq \phi_2' > \phi_2$. Since $M \in \mathcal{T}(\geq \phi_2')$, we have $\mathcal{T}_1 \subsetneq \mathcal{T}(\geq \phi_2') \subseteq \mathcal{T}_2$, so this either contradicts the fact that $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ is a cover relation or it contradicts the definition of ϕ_2. We conclude that $\phi_1 = \phi_2$, so that $\mathcal{T}_1^{\perp} \cap \mathcal{T}_2 = \mathsf{Wide}(=\phi_1)$ is a wide subcategory. □

We say a module M is **ϕ-stable** if M is a simple object of $\mathsf{Wide}(=\phi)$. If M is ϕ-stable and M' is ϕ' stable, and $\phi > \phi'$, then $\mathrm{Hom}(M, M') = 0$.

When $\mathsf{A} = \mathbf{rep}_n(\Pi_Q)$, we can think of a Bridgeland stability function Z as an $\mathbb{R}$-linear map $V \to \mathbb{C}$. Then $-\Re(Z)$ is a King stability condition as above, and $\mathcal{T}(\geq \frac{1}{2}) = \mathcal{T}(-\Re(Z))$. More generally, if $\phi \in (0,1)$ is a choice of phase, then $\theta = \Im(\exp(-i\phi\pi)Z)$ is a King stability condition and there are the identities $\mathcal{T}(\geq \phi) = \mathcal{T}(\theta)$, $\mathcal{T}(>\phi) = \mathcal{T}_*(\theta)$, and $\mathsf{Wide}(=\phi) = \mathsf{Wide}(\theta)$. With these identifications, Lemma 4.4.12 implies the following.

Lemma 4.4.17. *Let Z be a Bridgeland stability function on $\mathbf{rep}_n(\Pi_Q)$ so that each positive real root has a distinct phase. Then for each phase ϕ, there is at most one ϕ-stable real brick. If ϕ is the phase of a real root α, then there is exactly one ϕ-stable real brick.*

Lemma 4.4.18. *Let Z be a Bridgeland stability function on $\mathbf{rep}_n(\Pi_Q)$ so that each positive real root has a distinct phase. Let $I \subseteq (0,1)$ be the set of phases ϕ for which there exists a ϕ-stable real brick B_ϕ, and let $\{B_\phi\}_{\phi\in I}$ be the resulting brick sequence, ordered by decreasing phase. Then the map $I \to \Phi^+$ given by $\phi \mapsto \underline{\dim}\, B_\phi$ is a bijection, and the induced ordering of Φ^+ is a reflection order.*

Proof. Lemma 4.4.17 implies that the map $I \to \Phi^+$ is a bijection. Hence it is enough to check, for each $\phi_0 \in (0,1)$, that $\{\underline{\dim}\, B_\phi \mid \phi \geq \phi_0\}$ and $\{\underline{\dim}\, B_\phi \mid \phi > \phi_0\}$ are biclosed sets. Indeed, we have that

$$X \coloneqq \{\underline{\dim}\, B_\phi \mid \phi \geq \phi_0\} = \{\alpha \in \Delta^+_{\mathrm{re}} \mid \phi(\alpha) \geq \phi_0\} = \{\alpha \in \Delta^+_{\mathrm{re}} \mid \Im(\exp(-i\phi_0\pi)Z(\alpha)) \leq 0\}.$$

Since the function $\alpha \mapsto \Im(\exp(-i\phi_0\pi)Z(\alpha))$ is a linear functional on V, the set $X \subseteq \Delta^+_{\mathrm{re}}$ is separable. In particular, X is biclosed. The other case is similar. □

4.5 Torsion classes and biclosed sets

4.5.1 Imaginary bricks and forcing

Fix a quiver Q without loops and let Π_Q be the preprojective algebra. In this section, all bricks will be bricks in $\mathbf{rep}_n(\Pi_Q)$. We identify $\mathsf{Tors}(\Pi_Q)$ with $\mathrm{Pairs}_{\mathsf{Bricks}(\Pi_Q)}(\to)$ as in Proposition 4.4.4.

Lemma 4.5.1. *Let D be an imaginary brick. Then D does not directly force any real brick. If, furthermore, Q is an affine quiver, then D also does not force any other imaginary brick.*

Proof. Let M be a real brick. $\mathcal{T}(D)$ directly forces $\mathcal{T}(M)$ if and only if either

- $\mathcal{T}(D) \hookrightarrow \mathcal{T}(M)$ and $\mathcal{T}' \subsetneq \mathcal{T}(D)$ implies that $\mathcal{T}' \not\hookrightarrow \mathcal{T}(M)$, or

- $\mathcal{T}(M) \twoheadrightarrow \mathcal{T}(D)$ and if $\mathcal{T}' \hookrightarrow \mathcal{T}(D)$ and $\mathcal{T}' \neq \mathcal{T}(D)$ then $\mathcal{T}(M) \not\twoheadrightarrow \mathcal{T}(D)$.

We show the first case does not happen; the other case is similar. The first case occurs if and only if D is a submodule of M and no brick in $\mathcal{T}(D)$ other than D is a submodule of M. Equivalently, $0 \neq \Xi_D(M) \in \mathsf{Wide}(D)$. For a fixed $\delta, \alpha \in \Delta^+$, the statement "if D is a brick for Π_Q of dimension vector δ and M is a brick of dimension vector α, then $\Xi_D(M) = 0$ or $\Xi_D(M) \notin \mathsf{Wide}(D)$" is expressible in the first-order language of the field k. (Note that $\Xi_D(M) \in \mathsf{Wide}(D)$ if and only if either $\operatorname{Hom}(D, M) = 0$ or $D \subseteq M$ and $\Xi_D(M/D) \in \mathsf{Wide}(D)$; since the dimension of M is fixed, this recursive characterization can be translated into a first-order characterization.) By completeness of the theory of algebraically closed fields of a given characteristic, the statement is true when k is characteristic 0 if and only if it is true when $k = \overline{\mathbb{F}}_p$ for infinitely many primes p. For a given p, the statement is true for $k = \overline{\mathbb{F}}_p$ if and only if it is true for all $k = \mathbb{F}_q$ with q a power of p. So assume $k = \mathbb{F}_q$. We will show that if L is a finite quotient lattice of $\mathsf{Tors}(\Pi_Q)$ and D is an uncontracted imaginary brick of L so that $\dim D$ is maximal among the uncontracted imaginary bricks of L, then there is a quotient of L contracting D and no other bricks. Since $\mathsf{Tors}(\Pi_Q)$ is profinite by Lemma 4.4.6, this will imply that if D is an imaginary brick and M is a real brick then D does not force M, by taking a finite lattice quotient with D and M uncontracted and then contracting the imaginary bricks one at a time. This will then imply that D does not directly force M in $\mathsf{Tors}(\Pi_Q)$ and hence that the statement is true over $\mathbb{F}_q$.

Now let L be a finite lattice quotient of $\mathsf{Tors}(\Pi_Q)$, let D be an uncontracted imaginary brick with $\dim D$ maximal, and let M be a real brick. Let $\equiv$ be the equivalence relation on L so that $x \equiv y$ if and only if $x = y$ or x and y are related by a cover relation with brick label D. We need to check this is a complete lattice congruence. Let $\pi^{\downarrow}, \pi^{\uparrow} : \mathsf{L} \to \mathsf{L}$ be the maps sending an element x of L to the minimal (resp. maximal) element of the $\equiv$-equivalence class of x. (Each equivalence class is either a singleton or a cover relation, so this is well-defined.) We need to check that $\pi^{\downarrow}$ and $\pi^{\uparrow}$ are order-preserving. We handle the first, the other is similar. Since L is finite, we can check order-preservingness on cover relations. Assume $x \lessdot y$ is a cover relation. The nontrivial case is when $\pi^{\downarrow}(y) \lessdot y$. Let $\mathcal{Y}$ be the minimal element of $\mathsf{Tors}(\Pi_Q)$ mapping to y. Then there are torsion classes $\mathcal{X}$ and $\mathcal{P}$ mapping to x and $\pi^{\downarrow}(y)$ so that $\mathcal{X} \lessdot \mathcal{Y} \gtrdot \mathcal{P}$ in $\mathsf{Tors}(\Pi_Q)$. Then $[\mathcal{X} \wedge \mathcal{P}, \mathcal{Y}]_{\mathsf{Tors}(\Pi_Q)}$ is a wide interval. The corresponding wide subcategory is $\mathsf{Wide}(M, D)$. Hence there is a brick label-preserving isomorphism $[x \wedge \pi^{\downarrow}(y), y] \cong \mathsf{L}'$, where L' is the quotient of $\mathsf{Tors}(\mathsf{Wide}(D, M))$ contracting the bricks which are contracted in L.

Let $\alpha = \underline{\dim}\, M$ and $\delta = \underline{\dim}\, D$. We claim every brick of $\mathsf{Wide}(D, M)$ is either isomorphic to M or is an imaginary brick. To see this, assume N is a real brick distinct from M. In particular, $\underline{\dim}\, N$ is a real root. Then $t_\alpha \underline{\dim}\, N < \underline{\dim}\, N$, since $(\alpha, \underline{\dim}\, N) > 0$. The last inequality holds because $\underline{\dim}\, N$ is a positive combination of α and δ, and

$$\begin{aligned} (\alpha, \delta) &\leq 0 \\ (\delta, \delta) &\leq 0 \\ (\underline{\dim}\, N, \underline{\dim}\, N) &> 0. \end{aligned}$$

But this situation is impossible, since if N is a real brick distinct from M of minimal dimension, then $t_\alpha \underline{\dim}\, N = \alpha$, which implies $\underline{\dim}\, N = -\alpha$, an absurdity. Hence there are no real bricks in $\mathsf{Wide}(D, M)$ distinct from M. It follows that the only uncontracted bricks in L' are D and M, since every other brick of $\mathsf{Wide}(D, M)$ is imaginary of dimension larger than $\dim D$. We conclude that L' is isomorphic to a diamond poset and, furthermore, that the cover relation $x \gtrdot x \wedge \pi^{\downarrow}(y)$ is labeled by D. This means $\pi^{\downarrow}(x) = x \wedge \pi^{\downarrow}(y) \lessdot \pi^{\downarrow}(y)$, so $\pi^{\downarrow}$ is order-preserving. We deduce that $\equiv$ is a lattice congruence of L contracting D but contracting no other brick which is uncontracted in L.

To see that an imaginary brick D does not force imaginary bricks when Q is affine, the proof is similar. Instead of inductively removing imaginary bricks one at a time, one can use that $\mathsf{Wide}(D, M)$ is itself isomorphic to a diamond poset for any brick M with

$\mathrm{Hom}(D, M) = \mathrm{Hom}(M, D) = 0$. This fact follows from the identity

$$0 = (\underline{\dim}\, M, \underline{\dim}\, D) = \dim \mathrm{Hom}(M, D) - \dim \mathrm{Ext}(M, D) + \dim \mathrm{Hom}(D, M)$$
$$= -\dim \mathrm{Ext}(M, D).$$

□

As a result of Lemma 4.5.1, there is a quotient of $\mathsf{Tors}(\Pi_Q)$ contracting exactly the imaginary bricks. Denote the resulting lattice quotient by $\mathsf{Tors}_{\mathrm{re}}(\Pi_Q)$. Let Φ^+ be the positive real roots of the root system associated to Q. We will say $X \subseteq \Phi^+$ is a root poset order ideal if for any $\beta \in X$ and any $\alpha \in \Phi^+$ with $\beta - \alpha$ a positive combination of positive roots, we have that $\alpha \in X$. By Lemma 4.4.5, for each root poset ideal $X \subseteq \Phi^+$ of Φ^+, the set of bricks with dimension vector in X is forcing-open. Hence there is a quotient of $\mathsf{Tors}_{\mathrm{re}}(\Pi_Q)$ leaving uncontracted exactly those bricks with dimension vector in X. Denote the resulting quotient by $\mathsf{Tors}_X(\Pi_Q)$. We note that by Proposition 4.4.7, maximal chains in $\mathsf{Tors}_{\mathrm{re}}(\Pi_Q)$ correspond to maximal sequences of real bricks $\{B_i\}_{i\in I}$ with the property that $i < j$ implies $\mathrm{Hom}(B_i, B_j) = 0$. Similarly, maximal chains in $\mathsf{Tors}_X(\Pi_Q)$ correspond to maximal sequences of real bricks $\{B_i\}_{i\in I}$ with the property that $i < j$ implies $\mathrm{Hom}(B_i, B_j) = 0$ and so that $\underline{\dim}\, B_i \in X$ for all $i \in I$.

Lemma 4.5.2. *If $X \subseteq \Phi^+$ is a finite root poset order ideal, then $\mathsf{Tors}_X(\Pi)$ is a finite lattice. As a result, $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ is a profinite lattice.*

Proof. By [22, Theorem 5.1], there are finitely many real bricks of a given dimension vector. Hence if $X \subseteq \Phi^+$ is a finite root poset order ideal, then $\mathsf{Tors}_X(\Pi)$ has finitely many uncontracted bricks, so is thus a finite lattice. Any real root is contained in a finite root poset order ideal, so any real brick is uncontracted in some $\mathsf{Tors}_X(\Pi_Q)$ with X finite. It follows that $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ is profinite. □

We note also an unexpected consequence of Lemma 4.5.1.

Corollary 4.5.3. *Let Q be an affine quiver. Then $\mathsf{Tors}(\Pi_Q)$ is a profinite lattice.*

Proof. By Corollary 2.4.28, it is enough to show that every brick B is contained in a finite forcing-open subset of $\mathsf{Bricks}(\Pi_Q)$. Indeed, if B is real then the set of real bricks of dimension at most $\dim B$ is a finite forcing-open subset by Lemma 4.4.5, Lemma 4.5.1, and the fact (implied by [22, Theorem 5.1]) that there are finitely many isomorphism classes of real bricks of a given dimension. Similarly, if B is imaginary, then the set of bricks either equal to B or real of dimension at most $\dim B$ is a finite forcing-open subset. $\square$

4.5.2 From torsion classes to biclosed sets

In this section, fix a finite loopless quiver Q and set $\Pi := \Pi_Q$. Let $\mathcal{T} \in \mathsf{Tors}(\Pi)$ be a torsion class. Define

$$\underline{\dim}\,\mathcal{T} = \{\underline{\dim}\,M \mid M \in \mathcal{T} \text{ is a real brick}\}.$$

Similarly, for a torsion-free class $\mathcal{F}$, define

$$\underline{\dim}\,\mathcal{F} = \{\underline{\dim}\,M \mid M \in \mathcal{F} \text{ is a real brick}\}.$$

Then $\underline{\dim}\,\mathcal{T}$ is a subset of Φ^+. In this section, we will show that this subset is always biclosed; moreover, Φ^+ is the disjoint union of $\underline{\dim}\,\mathcal{T}$ and $\underline{\dim}(\mathcal{T}^{\perp})$.

First note that the function $\underline{\dim} : \mathsf{Tors}(\Pi) \to \mathcal{P}\Phi^+$ factors through the quotient map $\mathsf{Tors}(\Pi) \twoheadrightarrow \mathsf{Tors}_{\mathrm{re}}(\Pi)$. We will see in the next section that, under the hypothesis of a semidistributivitable order on Φ^+, the map $\underline{\dim} : \mathsf{Tors}_{\mathrm{re}}(\Pi) \to \mathcal{P}\Phi^+$ is a bijection onto the biclosed sets.

First we recast Lemma 4.4.12 in terms of $\mathsf{Tors}_{\mathrm{re}}(\Pi)$.

Lemma 4.5.4. *There exists a maximal chain in $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ with corresponding real bricks $\{B_i\}_{i \in I}$, so that $i \mapsto \underline{\dim}\,B_i$ is a bijection $I \to \Phi^+_{\mathrm{re}}$. Furthermore, the induced ordering of Φ^+_{re} is a reflection order.*

Proof. By Lemma 4.4.12, it is enough to construct a Bridgeland stability function Z so that each real root has a distinct phase. If we fix a functional $g : V \to \mathbb{R}$ so that $g(\Phi^+) > 0$, and let $f : V \to \mathbb{R}$ be a generic functional, then $f + ig$ will be such a function. Indeed, it is equivalent for the function $f/g : \Phi^+ \to \mathbb{R}$ to be injective, which imposes a countable set of conditions on f of the form $f(\mathbf{v}) \neq 0$. A generic choice of f will satisfy such conditions. $\square$

Lemma 4.5.5. *Let $X \subseteq \Phi^+_{\mathrm{re}}$ be a finite root poset order ideal. Then $\mathsf{Tors}_X(\Pi)$ is a polygonal lattice. As a result, $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ is pro-polygonal.*

Proof. Let $\overline{\mathcal{T}} \lessdot \overline{\mathcal{T}}_1$ and $\overline{\mathcal{T}} \lessdot \overline{\mathcal{T}}_2$ be cover relations in $\mathsf{Tors}_X(\Pi)$ with brick labels B_1, B_2. Then if $\mathcal{T}$ is the maximal element of $\mathsf{Tors}(\Pi)$ in the equivalence class $\overline{\mathcal{T}}$, then there exist $\mathcal{T}_1, \mathcal{T}_2$ in the classes $\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2$ so that $\mathcal{T} \lessdot \mathcal{T}_1, \mathcal{T}_2$ with brick labels B_1,B_2. The interval $[\mathcal{T}, \mathcal{T}_1 \vee \mathcal{T}_2]_{\mathsf{Tors}(\Pi)}$ is isomorphic to $\mathsf{Tors}(\mathsf{Wide}(B_1, B_2))$. Let X' be the set of roots in X which are in the span of $\underline{\dim}\, B_1, \underline{\dim}\, B_2$. Then the interval $[\overline{\mathcal{T}}, \overline{\mathcal{T}}_1 \vee \overline{\mathcal{T}}_2]_{\mathsf{Tors}_X(\Pi)}$ is isomorphic to $\mathsf{Tors}_{X'}(\mathsf{Wide}(B_1, B_2))$. We claim this lattice is a polygon. We check this by computing the two-acyclic factorization system $(\mathsf{Bricks}_{\mathrm{re}}(\mathsf{Wide}(B_1, B_2)), \to, \twoheadrightarrow, \hookrightarrow)$. If Ш is a finite set of the form $Ш = \{b_1, b_2\} \sqcup Ш_1 \sqcup Ш_2$ satisfying $b_1 \not\twoheadrightarrow Ш_1$, $b_2 \not\twoheadrightarrow Ш_2$, $Ш_1 \to b_1$, $Ш_2 \to b_2$, $b_1 \not\twoheadrightarrow b_2$, $b_2 \not\twoheadrightarrow b_1$, $b_1 \to Ш_2$, $b_2 \to Ш_1$, $Ш_1 \not\twoheadrightarrow b_2$, $Ш_2 \not\twoheadrightarrow b_1$, and that $Ш_1, Ш_2$ are both totally ordered by $\not\twoheadrightarrow$, then $\mathrm{Pairs}_Ш(\to)$ is a polygon. Write $\mathsf{Bricks} \coloneqq \mathsf{Bricks}_{\mathrm{re}}(\mathsf{Wide}(B_1, B_2))$ for the set of real bricks in $\mathsf{Wide}(B_1, B_2)$. We will show that the above setup holds for the decomposition $\mathsf{Bricks} = \{B_1, B_2\} \sqcup \mathcal{B}_1 \sqcup \mathcal{B}_2$, where $\mathcal{B}_1 \coloneqq \{B \in \mathsf{Bricks} \setminus \{B_2\} \mid B_1 \not\twoheadrightarrow B\}$ and $\mathcal{B}_2 \coloneqq \{B \in \mathsf{Bricks} \setminus \{B_1\} \mid B_2 \not\twoheadrightarrow B\}$. Then it will also hold for the restriction to the bricks with dimension vectors in X', which is a finite set; hence, $\mathsf{Tors}_{X'}(\mathsf{Wide}(B_1, B_2))$ is a polygon.

Recall that if A, B are bricks, then $A \to B$ means $\mathrm{Hom}(A, B) \neq 0$. Most of the relations in the setup above are easy to check. The difficulty is showing that $\not\twoheadrightarrow$ is a total order on $\mathcal{B}_1$ (and similarly for $\mathcal{B}_2$). We first check that the unique pair of real bricks A, B so that $A \not\twoheadrightarrow B$ and $B \not\twoheadrightarrow A$ is B_1, B_2. If $\{A, B\} \cap \{B_1, B_2\} \neq \varnothing$ then this is clear. Assume otherwise. Let $\beta_1 = \underline{\dim}\, B_1$ and $\beta_2 = \underline{\dim}\, B_2$. Without loss of generality, $t_{\beta_1} \underline{\dim}\, A < \underline{\dim}\, A$. A is in either $\mathbf{NoSub}_{B_1}$ or in $\mathbf{NoQuot}_{B_1}$; assume the first case (the other is similar). Then B_1 is a quotient of A, since A has composition factors B_1 and B_2 and is not isomorphic to B_2. If B_1 were a submodule of B, then there would be an arrow $A \to B$. Otherwise, $B \in \mathbf{NoSub}_{B_1}$.

Now we may apply the reflection (or spherical twist) functor $\sigma_{B_1}^{-1}$ across the simple module B_1 in $\mathsf{Wide}(B_1, B_2)$. Then there are no arrows between the two modules $\sigma_{B_1}^{-1}(A), \sigma_{B_1}^{-1}(B)$, and $\underline{\dim}\, \sigma_{B_1}^{-1}(A) = t_{\beta_1} \underline{\dim}\, A < \underline{\dim}\, A$. Hence by induction $\{\sigma_{B_1}^{-1}(A), \sigma_{B_1}^{-1}(B)\} = \{B_1, B_2\}$. But this contradicts the fact that $\sigma_{B_1}^{-1}(A), \sigma_{B_1}^{-1}(B) \in \mathbf{NoQuot}_{B_1}$. Hence $\{A, B\} = \{B_1, B_2\}$.

Let $A, B \in \{B_1, B_2\} \cup \mathcal{B}_1$ be distinct and not equal to B_1, B_2. We will now show that $A \to B$ if and only if $B \not\to A$. Indeed, we can induct on $\min\{\dim A, \dim B\}$; we will simultaneously show the claim for $\mathcal{B}_2$. Say $\dim A \leq \dim B$; the other case is similar. If A is B_1 then $B \to A$ and $A \not\to B$; if A is B_2 then $A \to B$ and $A \not\to B$. Otherwise, either $t_{\beta_1} \underline{\dim}\, A < \underline{\dim}\, A$ or $t_{\beta_2} \underline{\dim}\, A < \underline{\dim}\, A$. We handle the first case; the other is similar. Note that $\mathcal{B}_1 = \mathbf{NoSub}_{B_1} \setminus \{B_2\}$. Then since $A, B \in \mathbf{NoSub}_{B_1}$, there is an arrow $A \to B$ if and only if there is an arrow $\sigma_{B_1}^{-1}(A) \to \sigma_{B_1}^{-1}(B)$ and similarly for $B \to A$. Since $\underline{\dim}\, \sigma_{B_1}^{-1}(A) = t_{\beta_1} \underline{\dim}\, A < \underline{\dim}\, A$, by induction we know that $\sigma_{B_1}^{-1}(A) \to \sigma_{B_1}^{-1}(B)$ if and only if $\sigma_{B_1}^{-1}(B) \not\to \sigma_{B_1}^{-1}(A)$. This proves that $A \to B$ if and only if $B \not\to A$. So it remains to show that $\not\to$ is transitive: that if $A, B, C \in \mathcal{B}_1$ have $A \not\to B \not\to C$, then $A \not\to C$. Equivalently, if $C \to B$ and $B \to A$, then $C \to A$. We can similarly induct on $\min\{\dim A, \dim B, \dim C\}$, to reduce to the base case where $A, B, C \in \{B_1, B_2\} \cup \mathcal{B}_1$ or $A, B, C \in \{B_1, B_2\} \cup \mathcal{B}_2$, and $|\{A, B, C\} \cap \{B_1, B_2\}| = 1$, which is straightforward. $\square$

Lemma 4.5.6. *Let X be a set of vectors and let $\prec$ be a total order of X so that every initial section of $\prec$ is biclosed in X. Let $X' \subseteq X$ be the intersection of X with a 2-dimensional subspace so that $|X'| \geq 2$. Assume the elements of X' appear consecutively under $\prec$. Then defining $\prec'$ to be the ordering of X which coincides with $\prec$ on $X \setminus X'$ and which is the reverse of $\prec$ on X', every prefix of $\prec'$ is biclosed.*

Proof. Let B be the prefix consisting of vectors which are smaller than every element of X', and let B' be the prefix consisting of vectors which are not larger than every element of X'. Note that by the assumption of consecutivity, $B' = B \sqcup X'$. Let C be a prefix of $\prec'$. If $C \subseteq B$ or $C \supseteq B'$, then C is also a prefix of $\prec$, so we are done. Otherwise, there is some subset $C' \subseteq X'$ which is biclosed in X' so that $C = B \cup C'$. Let $\alpha, \beta, \gamma \in X$ have

$\gamma \in \mathrm{cone}(\alpha, \beta)$. We wish to check that $\{\alpha, \beta, \gamma\} \cap C$ is not $\{\alpha, \beta\}$ or $\{\gamma\}$. If $\alpha, \beta, \gamma \in X'$ then this holds since C' is biclosed in X'. Otherwise $\{\alpha, \beta, \gamma\} \cap X' = \varnothing$, which is biclosed, or else $|\{\alpha, \beta, \gamma\} \cap X'| = 1$. In the latter case, $\{\alpha, \beta, \gamma\} \cap C = \{\alpha, \beta, \gamma\} \cap B'$, so this is also biclosed. $\square$

Using polygonality, we can deform any two maximal chains into one another via rank 2 moves. This lets us deduce the following strengthening of Lemma 4.5.4.

Lemma 4.5.7. *For any maximal chain in* $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ *with corresponding real bricks* $\{B_i\}_{i \in I}$*, the map* $i \mapsto \underline{\dim}\, B_i$ *is a bijection* $I \to \Phi^+$ *and the induced ordering of* Φ^+ *is a reflection order.*

Proof. Let $\mathcal{C}$ be a maximal chain with real bricks $\{B_i\}_{i \in I}$. We will show, for each finite root poset order ideal $X \subseteq \Phi^+$, that if $\{B_i\}_{i \in I'} = \{B_i\}_{i \in I} \cap \mathsf{Tors}_X(\Pi)$, then $i \mapsto \underline{\dim}\, B_i$ is a bijection from I' to X and any prefix of the induced ordering of X is a biclosed subset of X. By Lemma 4.5.4, Lemma 4.5.5, and the fact that chains in a polygonal lattice are connected by polygon moves [45, Lemma 9-6.3], it is enough to check that this property preserved under polygon moves. From the proof of Lemma 4.5.5, each polygon in $\mathsf{Tors}_X(\Pi)$ is isomorphic to $\mathsf{Tors}_{X'}(\mathsf{Wide}(B_1, B_2))$ for X' the restriction of X to the rank 2 subsystem of Φ spanned by $\underline{\dim}\, B_1, \underline{\dim}\, B_2$. We can construct the two chains of this polygon using Bridgeland stability functions as in Lemma 4.5.4; in particular, if Z is a generic stability function then $-\Re(Z) + i\Im(Z)$ is also a generic stability function, and the resulting order of X' is reversed. Hence a polygon move preserves the bijectivity of $i \mapsto \underline{\dim}\, B_i$, and the resulting operation on the total order of Φ^+ satisfies the hypothesis of Lemma 4.5.6. Hence every prefix of the order remains biclosed. $\square$

The following resolves and strengthens Conjecture 4.4.11.

Corollary 4.5.8. *Let* B_1 *and* B_2 *be real bricks so that* $\underline{\dim}\, B_1 = \underline{\dim}\, B_2$*. Then* $\mathrm{Hom}(B_1, B_2) \neq 0$*.*

Proof. Let B_1, B_2 be real bricks with $\mathrm{Hom}(B_1, B_2) = 0$. Then by Proposition 4.4.7 there is a maximal chain in $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ so that the real brick sequence $\{B_i\}_{i \in I}$ contains $\{B_1, B_2\}$. By Lemma 4.5.7, $\underline{\dim}\, B_1 \neq \underline{\dim}\, B_2$. $\square$

Theorem 4.5.9. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair for a preprojective algebra Π_Q with real root system Φ_Q. Then*

$$\underline{\dim}\, \mathcal{T} \sqcup \underline{\dim}\, \mathcal{F}$$

is a partition of Φ^+ into two biclosed sets.

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then $\mathcal{T}$ is contained in some maximal chain of torsion classes with associated brick sequence $\{B_i\}_{i \in I}$. There is a partition of I into a prefix and suffix $I_1 \sqcup I_2$ so that $\{B_i\}_{i \in I_1} \subseteq \mathcal{T}$ and $\{B_i\}_{i \in I_2} \subseteq \mathcal{F}$. The set $\{\underline{\dim}\, B_i\}_{i \in I_1}$ is independent of the choice of maximal chain in $[0, \mathcal{T}]$ used to compute it, since any two chains are related by polygon moves as in the proof of Lemma 4.5.7. Hence

$$\{\underline{\dim}\, B_i\}_{i \in I_1} = \{\underline{\dim}\, M \mid M \in \mathcal{T} \text{ is a real brick}\} = \underline{\dim}\, \mathcal{T}$$

and, similarly,

$$\{\underline{\dim}\, B_i\}_{i \in I_2} = \{\underline{\dim}\, M \mid M \in \mathcal{F} \text{ is a real brick}\} = \underline{\dim}\, \mathcal{F},$$

so by Lemma 4.5.7 $\underline{\dim}\, \mathcal{T}$ is a biclosed set and is the complement of $\underline{\dim}\, \mathcal{F}$. $\square$

4.5.3 The isomorphism

In this subsection, we will assume Conjecture 3.3.4 for the main result. By Theorem 3.3.7, the results are unconditional for finite and affine root systems.

Lemma 4.5.10. *Let $X \subseteq \Phi^+$ be a finite root poset order ideal which is also a semidistributivitable order ideal and $\{B_i\}_{i \in I}$ be the brick sequence associated to a maximal chain in $\mathsf{Tors}_X(\Pi)$. If X' is the intersection of X with a full rank 2 subsystem of Φ, and $\{i \in I \mid$*

$\underline{\dim}\, B_i \in X'\}$ *is an order convex subset of* I*, then there is a maximal chain in* $\mathsf{Tors}_X(\Pi)$ *with brick sequence* $\{B_i\}_{i\in I'}$ *coinciding with* $\{B_i\}_{i\in I'}$ *except for those* B_i *with* $\underline{\dim}\, B_i \in X'$*; the sequences* $\{\underline{\dim}\, B_i\}_{i\in I} \cap X'$ *and* $\{\underline{\dim}\, B_i\}_{i\in I'} \cap X'$ *are reverses of one another.*

Proof. Let $\overline{\mathcal{T}}_1$ be the maximal element of the chain not containing any B_i with $\underline{\dim}\, B_i \in X'$, and let $\overline{\mathcal{T}}_2$ be the minimal element of the chain containing $\{B_i \mid i \in I,\ \underline{\dim}\, B_i \in X'\}$. Let $\mathcal{T} \in \mathsf{Tors}(\Pi)$ be the maximal torsion class projecting to $\overline{\mathcal{T}}_1$, and let $\mathcal{T}_2 \in \mathsf{Tors}(\Pi)$ be the minimal torsion class projecting to $\overline{\mathcal{T}}_2$. Then there is an upper cover of $\mathcal{T}_1$ with brick label B_1 and a lower cover of $\mathcal{T}_2$ with brick label B_2, so that $\{\underline{\dim}\, B_1, \underline{\dim}\, B_2\}$ are the simple roots of the rank 2 root subsystem spanned by X'. Hence $\mathcal{T}_1^{\perp} \cap \mathcal{T}_2$ contains $\mathsf{Wide}(B_1, B_2)$. For any real brick $B \in \mathcal{T}_1^{\perp} \cap \mathcal{T}_2$ with $\underline{\dim}\, B \in X$, there is a saturated chain from $\overline{\mathcal{T}}_1$ to $\overline{\mathcal{T}}_2$ using B as a brick label. Since chains are connected by polygon moves, and there is at most one polygon move involving bricks with dimension vectors in a rank 2 subsystem X', we must have $B \in \mathsf{Wide}(B_1, B_2)$. Hence $[\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2]$ is isomorphic to $\mathsf{Tors}_{X'}(\mathsf{Wide}(B_1, B_2))$. In particular, it is a polygon, and performing the corresponding polygon move to our chain proves the claim. □

Given a maximal chain $\mathcal{C}$ in $\mathsf{Tors}_{\mathrm{re}}(\Pi)$, write $\underline{\dim}\, \mathcal{C}$ for the maximal chain $\{\underline{\dim}\, \mathcal{T} \mid \mathcal{T} \in \mathcal{C}\}$ in $\mathrm{Bic}(\Phi^+_{\mathrm{re}})$.

Lemma 4.5.11. *If* $\mathcal{C}_1, \mathcal{C}_2$ *are two maximal chains in* $\mathsf{Tors}_{\mathrm{re}}(\Pi)$ *so that* $\underline{\dim}\, \mathcal{C}_1 = \underline{\dim}\, \mathcal{C}_2$*, then* $\mathcal{C}_1 = \mathcal{C}_2$*.*

Proof. It is enough to show that for all finite root poset order ideals $X \subseteq \Phi^+$, and $\mathcal{C}_1, \mathcal{C}_2$ are maximal chains in $\mathsf{Tors}_X(\Pi)$ with $\underline{\dim}\, \mathcal{C}_1 = \underline{\dim}\, \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2$. By Lemma 4.5.5 and Lemma 4.5.10, we can assume that $\mathcal{C}_1$ comes from a stability condition. Hence it is enough to show that if $\mathcal{C}$ is a maximal chain in $\mathsf{Tors}_X(\Pi)$ and $\underline{\dim}\, \mathcal{C}$ is the ordering prescribed by a stability function Z, then $\mathcal{C}$ is the chain prescribed by Z as in Lemma 4.4.17. Equivalently, it is enough to show if $\{B_i\}_{i\in I}$ is the brick sequence associated to $\mathcal{C}$, then each B_i is stable under Z. Let B_i be the brick label of the cover relation $\mathcal{T}_1 \lessdot \mathcal{T}_2$ in $\mathsf{Tors}_X(\Pi)$. By induction on the height of $\mathcal{T}_1$ in $\mathcal{C}$, $\mathcal{T}_1$ is the restriction of the torsion class $\mathcal{T}(> \phi(B_i))$. If there were a destabilizing submodule of B_i, then there would be a destabilizing brick submodule $B' \subseteq B_i$ with $B' \in \mathcal{T}(> \phi(B_i))$. By Lemma 4.5.1, there would be some real brick $B'' \subseteq B_i$ with $B' \in \mathcal{T}(> \phi(B_i))$. Hence $B'' \in \mathcal{T}_1$, contradicting that $B_i \in \mathcal{T}_1^{\perp}$. We deduce B_i is stable, so $\mathcal{T}_2$ is the restriction of $\mathcal{T}_2(\geq \phi(B_i))$. By induction, B_i is stable for all $i \in I$. □

Theorem 4.5.12. *Assume Conjecture 3.3.4 and that the semidistributivitable ordering refines the root poset. Then* $\underline{\dim} : \mathsf{Tors}_{\mathrm{re}}(\Pi) \to \mathrm{Bic}(W)$ *is an order isomorphism.*

Proof. It is enough to check for a finite suitable order ideal $X \subseteq \Phi^+$ that $\underline{\dim} : \mathsf{Tors}_X(\Pi) \to \mathrm{Bic}(X)$ is an order isomorphism. Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathsf{Tors}_X(\Pi)$ have $\underline{\dim}\,\mathcal{T}_1 = \underline{\dim}\,\mathcal{T}_2$. Since $\mathrm{Bic}(X)$ is polygonal, any two maximal chains in $[\varnothing, \underline{\dim}\,\mathcal{T}_1]_{\mathrm{Bic}(X)}$ are related by polygon moves. In particular, if $\mathcal{C}$, $\mathcal{D}$ are maximal chains in $[0, \mathcal{T}_1]_{\mathsf{Tors}_X(\Pi)}$ and $[0, \mathcal{T}_2]_{\mathsf{Tors}_X(\Pi)}$, respectively, then $\underline{\dim}\,\mathcal{C}$ and $\underline{\dim}\,\mathcal{D}$ are maximal chains in $[\varnothing, \underline{\dim}\,\mathcal{T}_1]_{\mathrm{Bic}(X)}$, so are related by applying a sequence of polygon moves to $\underline{\dim}\,\mathcal{C}$. By Lemma 4.5.10, each such move lifts to a polygon move in $[0, \mathcal{T}_1]_{\mathsf{Tors}_X(\Pi)}$. Hence we can assume $\mathcal{C}$ satisfies $\underline{\dim}\,\mathcal{C} = \underline{\dim}\,\mathcal{D}$. Similarly we can find chains $\mathcal{C}'$ and $\mathcal{D}'$ in $[\mathcal{T}_1, \top]_{\mathsf{Tors}_X(\Pi)}$ and $[\mathcal{T}_2, \top]_{\mathsf{Tors}_X(\Pi)}$ so that $\underline{\dim}\,\mathcal{C}' = \underline{\dim}\,\mathcal{D}'$. Now Lemma 4.5.11 implies that $\mathcal{C} \cup \mathcal{C}' = \mathcal{D} \cup \mathcal{D}'$, so $\mathcal{T}_1 = \mathcal{T}_2$.

By a similar argument, any maximal chain in $\mathrm{Bic}(X)$ is of the form $\underline{\dim}\,\mathcal{C}$ for a (unique) maximal chain $\mathcal{C}$ of $\mathsf{Tors}_X(\Pi)$. In particular, the map $\underline{\dim} : \mathsf{Tors}_X(\Pi) \to \mathrm{Bic}(X)$ is bijective. Furthermore, if $\underline{\dim}\,\mathcal{T}_1 \subseteq \underline{\dim}\,\mathcal{T}_2$, then there is some maximal chain in $\mathrm{Bic}(X)$ containing $\{\underline{\dim}\,\mathcal{T}_1, \underline{\dim}\,\mathcal{T}_2\}$, which is the image of a maximal chain $\mathcal{C}$ of $\mathsf{Tors}_X(\Pi)$. By injectivity of $\underline{\dim}$, the chain $\mathcal{C}$ must contain $\mathcal{T}_1$ and $\mathcal{T}_2$, so $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Hence $\underline{\dim} : \mathsf{Tors}_X(\Pi) \to \mathrm{Bic}(X)$ is an order isomorphism. □

4.5.4 Application to cluster algebras

In this section we apply Theorem 4.5.12 to the theory of cluster algebras. The **ordered exchange graph** of a cluster algebra A is a directed graph with nodes corresponding to the seeds of A and edges between seeds that are related by a mutation, with the orientation determined by Keller's red-to-green rule. We refer to [18] for more details and references.

Theorem 4.5.13. *Let Q be an affine quiver (including the oriented n-cycle), and let A_Q be the cluster algebra (with principle coefficients) associated to Q. Then there is a complete lattice congruence $\equiv_Q$ of* $\mathrm{Bic}(W)$ *so that the ordered exchange graph of A_Q embeds as a subgraph of the Hasse diagram of* $\mathrm{Bic}(W)/{\equiv_Q}$.

Proof. Let Q be an affine quiver and A_Q the principal coefficients cluster algebra associated to Q. We wish to show that there is a lattice quotient L of $\mathrm{Bic}(W)$ so that the Hasse diagram of L contains the ordered exchange graph of A_Q. The key is that there is a known lattice quotient L of $\mathsf{Tors}(\Pi_Q)$ which has the property we desire. If Q is acyclic, then we set $\Lambda := k[Q]$ to be the path algebra of Q. If Q is the oriented cycle, then we set Λ to be the quotient of $k[Q]$ which is a cluster-tilted algebra of type D. This is the quotient by the ideal generated by paths of length $n-1$. In both cases, Λ is a quotient algebra of Π_Q, so there is a quotient map $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{Tors}(\Lambda)$. When Λ is a path algebra of an acyclic quiver or a cluster-tilted algebra, then it is known (see, e.g., [18, Section 4.5]) that the Hasse diagram of $\mathsf{Tors}(\Lambda)$ contains the ordered exchange graph of A_Q in the connected component of the bottom element.

Our goal is now to prove that there is a common quotient of $\mathsf{Tors}(\Lambda)$ and $\mathrm{Bic}(W)$. We define L to be the quotient of $\mathsf{Tors}(\Lambda)$ given by contracting all imaginary bricks of Λ. By Lemma 4.5.1, no other bricks are contracted in this quotient. Then $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{L}$ factors through the quotient $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathrm{Bic}(W)$, since all bricks contracted in the second quotient are also contracted in the first.

We now wish to show that the quotient $\mathsf{Tors}(\Lambda) \twoheadrightarrow \mathsf{L}$ maps the ordered exchange graph isomorphically onto its image. Since each edge in the exchange graph has a brick label which is preserved by the quotient for uncontracted edges, the only way this could fail is if the brick label of some edge of the exchange graph is contracted in the quotient. In other words, we wish to show that the bricks labeling the edges of the exchange graph are all real bricks. Now, if M is a brick labeling an edge of the exchange graph, then $\underline{\dim}(M)$ is a c-vector of the cluster algebra [18, $\mathsf{f\text{-}tors} \to \mathsf{int\text{-}t\text{-}str} \to \mathsf{2\text{-}smc} \to \mathsf{c\text{-}mat}$]. It is known that the c-vectors of acyclic cluster algebras [18, Theorem 3.23] and of the oriented cycle cluster algebra (e.g.,

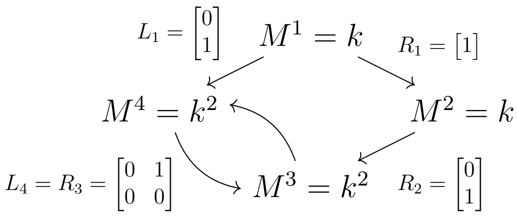


Figure 4.1: A module for Π_Q. The unlabeled linear map is 0.

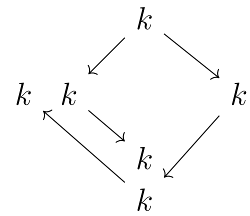


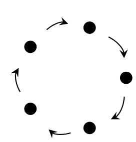
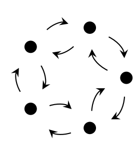
Figure 4.2: A string module, equivalent to the module in Figure 4.1.

[3]) are real roots. Hence all brick labels must be real, and $\mathsf{Tors}(\Lambda) \twoheadrightarrow \mathsf{L}$ restricts to an isomorphism on the ordered exchange graph. □

Remark 4.5.14. The lattice L constructed in the theorem could rightfully be called a **Cambrian lattice** for affine quivers. The theory of Cambrian lattices for finite Coxeter groups is well-understood and is a rich source of combinatorics, but the existence of these lattices in infinite type has been conjectural [47] until now and motivated our joint works with David Speyer [7, 8]. In the case where Q is the oriented cycle, we call the resulting lattice the **affine Tamari lattice** and have developed its theory with Colin Defant in [3].

4.5.5 Type $\widetilde{A}_{n-1}$

In this section we make explicit the combinatorics governing the oriented cycle quiver Q with n vertices, which is the affine quiver of type $\widetilde{A}_{n-1}$. The oriented cycle Q and its double quiver $\overline{Q}$ are shown below.

$Q =$  $\overline{Q} =$ 

The data of a $k[\overline{Q}]$ module M is equivalently the data of for each $i \in \{0, \ldots, n-1\}$: a vector space M^i, a linear map $M^i \xrightarrow{R_i} M^{i+1}$, and a linear map $M^{i-1} \xleftarrow{L_i} M^i$ (here we take the indices to be cyclic modulo n). These data define a module for Π_Q if and only if $R_i L_{i+1} - L_i R_{i-1} = 0$ for all $i \in \{0, \ldots, n-1\}$. See Figure 4.1 for an example of a Π_Q-module.

We will be particularly interested in the **string modules** of $k[\overline{Q}]$. These are modules that can be depicted as in Figure 4.2. We interpret this picture in the following way: each

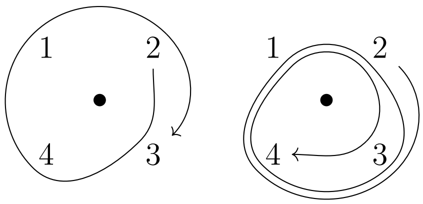


Figure 4.3: Two real noncrossing arc data. The arc on the left has TITO $[0,1][\underline{2,3}]$ and the arc on the right has TITO $[\overline{7,12,2,5}]$.

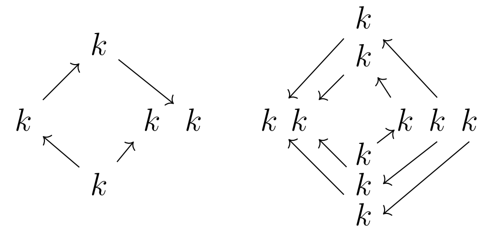


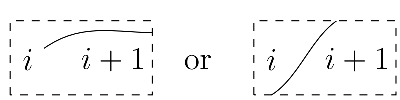
Figure 4.4: The string modules associated to the arc data in Figure 4.3.

copy of k over a vertex i corresponds to a basis vector of M^i. The map R_i sends a basis vector e in M^i to the basis vector of M^{i+1} that the clockwise-leaving arrow from e points to, or to 0 if there is no clockwise-leaving arrow from e. We define L_i similarly. Hence the modules depicted in Figures 4.1 and 4.2 are the same. Not every string module is a spherical module, but we will show below that every spherical module is a string module.

String modules can be encoded using the arc data from Section 3.4. Given an arc datum $\mathfrak{a} = (a, b, L, R)$, define a string module $\Pi_\mathfrak{a}$ with $\Pi_\mathfrak{a}^i$ spanned by basis vectors $\{e_j \mid a \le j < b,\ j \equiv i \bmod n\}$. If $j \in L$ then put an arrow from e_j to e_{j+1}, and if $j \in R$ then put e_{j+1} to e_j. Examples are shown in Figure 4.4.

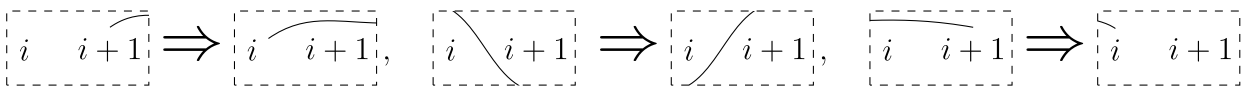
Lemma 4.5.15. *The map $J \mapsto \Pi_{\mathfrak{a}(J)}$ is a bijection $\mathsf{JIrr}^c(\mathrm{Bic}(W)) \xrightarrow{\sim} \mathsf{Sph}(\Pi_Q)$.*

Proof. By Proposition 3.4.9, it is enough to verify that the map $\mathfrak{a} \mapsto \Pi_\mathfrak{a}$ is a bijection from real noncrossing arc data to $\mathsf{Sph}(\Pi_Q)$. First, we note that $\mathfrak{a} \mapsto \Pi_\mathfrak{a}$ is an injective map, since we can recover the arc data $\mathfrak{a}$ from $\Pi_\mathfrak{a}$ via, e.g., a composition series. It remains to verify that $\Pi_\mathfrak{a}$ is in fact spherical, and that every spherical module arises in this way. A straightforward verification shows that $\Pi_\mathfrak{a}$ has S_i as a quotient if and only if a subarc of $\mathfrak{a}$ matches one of the following two patterns:



. (∗)

A less straightforward computation shows that if $\mathfrak{a}$ is a real noncrossing arc, and $\Pi_\mathfrak{a}$ is in $\mathbf{NoQuot}_i$, then $\sigma_i(\Pi_{\mathfrak{a}}) = \Pi_{\mathfrak{a}'}$ where $\mathfrak{a}'$ is another real noncrossing arc. We can construct $\mathfrak{a}'$ by imagining putting a finger on each of the marked points i and $i+1$ in the plane and twisting 180° counter-clockwise. During this twist, we deform $\mathfrak{a}$ so that it avoids crossing any marked points. If one of i or $i+1$ is an endpoint of $\mathfrak{a}$, then we also move that endpoint. After relabeling i and $i+1$ so they appear in their original order, the resulting arc is the new arc datum $\mathfrak{a}'$. This process translates to several local moves (applied separately to each segment of $\mathfrak{a}$ which passes through the box):



By [22, Theorem 4.3], it follows that for any $\Pi_{\mathfrak{a}} \in \mathbf{NoQuot}_i$ which is a real brick, we have that $\Pi_{\mathfrak{a}'}$ is a real brick. Similar remarks apply to σ_i^-, which acts by twisting 180° clockwise.

Given any shard arc $\mathfrak{a}$, we can perform a 180° twist involving one of the endpoints of $\mathfrak{a}$ to make a shorter shard arc $\mathfrak{a}'$. Without loss of generality, this twist is counter-clockwise. Then $\mathfrak{a}$ must avoid the patterns $(*)$ or the result of the twist would not be a shard arc. But then $\Pi_{\mathfrak{a}}$ is in $\mathbf{NoQuot}_i$, so it follows that $\Pi_{\mathfrak{a}} = \sigma_i^- \sigma_i \Pi_{\mathfrak{a}} = \sigma_i^- \Pi_{\mathfrak{a}'}$. By induction $\Pi_{\mathfrak{a}'}$ is a real brick, hence so is $\Pi_{\mathfrak{a}}$. This shows that the image of $\mathfrak{a} \mapsto \Pi_{\mathfrak{a}}$ consists of spherical modules.

To show surjectivity, let M be any real brick. Then $M = \sigma_i M'$ or $M = \sigma_i^- M'$ for some real brick $M' \in \mathbf{NoQuot}_i$ (resp. $M' \in \mathbf{NoSub}_i$) of smaller dimension [22, Theorem 5.1]. Without loss of generality, $M = \sigma_i M'$. By induction on dimension M' is of the form $\Pi_{\mathfrak{a}}$ for some shard arc $\mathfrak{a}$. Then $\sigma_i \Pi_{\mathfrak{a}} = \Pi_{\mathfrak{a}'}$ for some shard arc $\mathfrak{a}'$. We conclude that $\mathfrak{a} \mapsto \Pi_{\mathfrak{a}}$ is surjective. □

Remark 4.5.16. Under homological mirror symmetry, the operation of twisting an arc 180° corresponds to applying a *Dehn twist* around a Lagrangian sphere.

Corollary 4.5.17. *Every real brick for* Π_Q *is a shard module.*

Proof. Each shard module M is associated to a stability domain $\mathsf{Stab}(M)$, which is a cone, called a *shard*, in the $\widetilde{A}_{n-1}$ Coxeter arrangement. By [4], the shards are parametrized by arc data. This commutes with the parametrization above; since every real brick has an associated arc datum, every brick is a shard module. □

Chapter 5

Applications to Bruhat order

In this chapter, we will prove several results on the Bruhat order of a Coxeter group. The results in the third section are joint with Christian Gaetz, and the results in the first two sections were motivated by our discussions. We recall the definition of Bruhat order.

Definition 5.0.1. Let (W, S) be a Coxeter system with reflections $T = \{wsw^{-1} \mid w \in W,\ s \in S\}$. The **Bruhat graph** of W is the labeled directed graph on W with an edge $x \to y$ labeled by $t \in T$ whenever $tx = y$ and $\ell(x) < \ell(y)$. The **Bruhat order** on W is the partial order putting $x \leq y$ whenever there is a path $x \to \cdots \to y$ in the Bruhat graph. The **length** of an edge $x \to y$ is $\ell(y) - \ell(x)$.

We let Γ_W denote the Bruhat graph of W, and write $\Gamma_{u,v}$ for the induced subgraph on the elements of the Bruhat order interval $[u, v]$. Here we collect some important properties of Bruhat order.

Proposition 5.0.2. *Let W be a Coxeter group and $u, v \in W$. Then the Bruhat order interval $[u, v]$ is finite and graded by $x \mapsto \ell(x) - \ell(u)$. There is a cover relation $x \lessdot y$ in $[u, v]$ if and only if there is an edge $x \to y$ in $\Gamma_{u,v}$ of length one.*

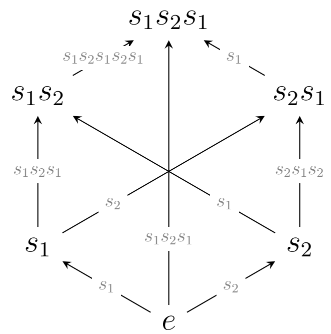


Figure 5.1: The Bruhat graph $\Gamma_{e,s_1s_2s_1}$ for a Coxeter group (W, S) where $s_1, s_2 \in S$ are such that s_1s_2 has order ≥ 3. Edge labels are shown in gray.

5.1 EL-labeling Bruhat order

Let P be a finite poset, viewed as a graph whose edges are cover relations. Let T be a set with a total order $\prec$. Given an edge labeling of P by the elements of T, we write $T(x, x')$ for the edge label of the cover relation $x \lessdot x'$. Such an edge labeling defines a set of *increasing chains* between $x, y \in P$ with $x < y$, which are maximal chains $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y$ so that $T(x_i, x_{i+1}) \prec T(x_{i+1}, x_{i+2})$ for all i. Then an edge labeling of P by elements of T is said to be an **EL-labeling** if the following two properties hold, for every $x, y \in P$ with $x < y$:

- There is a unique increasing chain from x to y, and
- The unique increasing chain from x to y is the lexicographically minimal chain; i.e., if $x_0 \lessdot \cdots \lessdot x_k$ is an increasing chain, and $x_i \lessdot z \leq x_k$, then $T(x_i, x_{i+1}) \preceq T(x_i, z)$.

By Definition 5.0.1, each cover relation $x \lessdot y$ in Bruhat order has a corresponding edge $x \to y$ in the Bruhat graph. Hence we can label the cover relations of Bruhat order by setting $T(x, y)$ to be the label of the edge $x \to y$. The following theorem is [31, Proposition 4.3]. Recall the notion of *reflection orders* from Definition 3.0.4.

Theorem 5.1.1. *Let W be a Coxeter group with reflections T and let $\prec$ be a reflection order on T. Then for any $u, v \in W$, the edge-labeling of the poset $[u, v]$ by reflections (ordered by $\prec$) is an EL-labeling.*

Here we give a new proof, assuming Conjecture 3.3.4, of an important lemma used in the proof of Theorem 5.1.1. We remark that prior proofs of the lemma (for all Coxeter groups) use Hecke algebras [31] or a complicated recurrence [12, 30] derived from Hecke algebras, while our proof naturally uses extended weak order. The lemma is the only non-combinatorial input to the proof of Theorem 5.1.1, which can then be finished by a more straightforward recurrence that we describe below.

Fix a Coxeter group W and elements u, v with $u \leq v$. If $\prec$ is a total ordering of $\{t \in T \mid \exists x, y \in [u, v],\ x \xrightarrow{t} y\}$ (for instance, the restriction of a reflection order), then let $r_{\prec}(u, v)$ denote the number of increasing chains from u to v.

Lemma 5.1.2. *Let $\prec_1$ and $\prec_2$ be two reflection orders. Assume Conjecture 3.3.4. Then $r_{\prec_1}(u, v) = r_{\prec_2}(u, v)$.*

Proof. Let Φ be a root system for W and fix a suitable order on Φ^+ satisfying Conjecture 3.3.4. Let $I \subseteq \Phi^+$ be a finite suitable order ideal containing $\{\alpha_t \mid \exists x, y \in [u, v],\ x \xrightarrow{t} y\}$. Each reflection order determines a maximal chain in $\mathrm{Bic}(\Phi^+)$ (its collection of initial sections), which restricts to a maximal chain in $\mathrm{Bic}(I)$ (by Lemma 3.2.4). Let $T' = \{t \in T \mid \alpha_t \in I\}$. Then each maximal chain of $\mathrm{Bic}(I)$ is the collection of initial sections of some unique total ordering of T'. Hence it is enough to show, for any total orderings $\prec_1, \prec_2$ of T' arising from maximal chains of $\mathrm{Bic}(I)$, that $r_{\prec_1}(u, v) = r_{\prec_2}(u, v)$.

By Theorem 3.3.9, any two maximal chains of $\mathrm{Bic}(I)$ are related by a sequence of rank 2 flips. Hence, it is enough to show that if $\prec_1$ and $\prec_2$ are associated to two maximal chains that differ by a rank 2 flip, then $r_{\prec_1}(u, v) = r_{\prec_2}(u, v)$. So assume $\prec_1$ and $\prec_2$ are two such orders, associated to maximal chains $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathrm{Bic}(I)$. Let

$$\mathcal{A}_2 = \{\alpha \in \Phi^+ \mid \exists B_1 \in \mathcal{C}_1, B_2 \in \mathcal{C}_2,\ |B_1| = |B_2| \text{ and } \alpha \in B_1 \Delta B_2\}$$

be the full rank 2 subset of I on which $\mathcal{C}_1$ and $\mathcal{C}_2$ differ. Set $T_2 = \{t \in T \mid \alpha_t \in \mathcal{A}_2\}$ and let W' be the reflection subgroup of W generated by T_2 (necessarily of rank 2). Then the ordering of T_2 induced by $\prec_1$ is the reverse of the ordering induced by $\prec_2$, and both orders put T_2 consecutively within T'. Let $c = (u = x_0 \lessdot x_1 \cdots \lessdot x_{k-1} \lessdot x_k = v)$ be a $\prec_1$-increasing chain. If c is also a $\prec_2$-increasing chain, then we set $\mu(c) = c$. Otherwise, since $\prec_1$ and $\prec_2$ differ by reversing a consecutive subset of T', there is some saturated subchain $x_i \lessdot \cdots \lessdot x_j$ with edge labels $t_i, \ldots, t_{j-1}$ so that $t_{j-1} \prec_2 t_{j-2} \prec_2 \cdots \prec_2 t_i$ and the remaining edge labels $t_0, t_1, \ldots, t_{i-1}, t_{j+1}, \ldots, t_{k-1}$ are in increasing order under both $\prec_1$ and $\prec_2$. Equivalently, $t_i, \ldots, t_{j-1}$ is exactly the set of edge labels of c which are in the set T_2. Since $x_i \lessdot \cdots \lessdot x_j$ is a maximal chain of the interval $[x_i, x_j]$, and its edge labels come from T_2, by [29, Theorem 1.4] there is some $x'_i, x'_j \in W'$ with an edge-label preserving isomorphism $\phi : [x'_i, x'_j]_{W'} \xrightarrow{\sim} [x_i, x_j]$. By inspection, $x'_i \lessdot \phi^{-1}(x_{i+1}) \lessdot \cdots \lessdot x'_j$ is the unique $\prec_1$-increasing chain in $[x'_i, x'_j]_{W'}$. There is also a unique $\prec_1$-decreasing chain in $[x'_i, x'_j]_{W'}$, call it $y'_i \lessdot \cdots \lessdot y'_j$, which is necessarily the unique $\prec_2$-increasing chain. In this case, we define $\mu(c)$ to be the chain

$$x_0 \lessdot \cdots \lessdot x_{i-1} \lessdot \phi(y'_i) \lessdot \cdots \lessdot \phi(y'_j) \lessdot x_{j+1} \lessdot \cdots \lessdot x_k.$$

In both cases, $\mu(c)$ is a $\prec_2$-increasing chain. The assignment $c \mapsto \mu(c)$ is a bijection from the $\prec_1$-increasing chains of $[u, v]$ to the $\prec_2$-increasing chains of $[u, v]$. Hence $r_{\prec_1}(u, v) = r_{\prec_2}(u, v)$. $\square$

Remark 5.1.3. If W is a finite Coxeter group, then the suitable order ideal used in Lemma 5.1.2 can be taken to be all of Φ^+. Then Theorem 3.3.9 reduces to Matsumoto's theorem for reduced words of Coxeter groups, so our proof is elementary in that case.

Remark 5.1.4. Rather than counting increasing chains from u to v in Bruhat order, we could count **increasing paths** from u to v in the Bruhat graph. Increasing chains coincide with increasing paths of length $\ell(v) - \ell(u)$. Essentially the same proof as that of Lemma 5.1.2 shows that the number of increasing paths from u to v of length k is independent of the choice of reflection order used to compute it. This is another result of Dyer, who used it to give a description of *Kazhdan–Lusztig R-polynomials* using reflection orders [30].

The rest of Dyer's proof

In this section we complete the proof of Theorem 5.1.1. We split it into two lemmas, the proofs of which may be extracted from Dyer's work.

Lemma 5.1.5. *The lexicographically minimal chain of* $[u, v]$ *is an increasing chain.*

Proof. Let $c = (u = x_0 \lessdot \cdots \lessdot x_k = v)$ be the lexicographically minimal chain. Each interval $[x_i, x_{i+2}]$ is a diamond. By [29, Theorem 1.4], there is a dihedral reflection subgroup W' of W, elements $x'_i, x'_{i+2} \in W'$, and an edge-label preserving isomorphism $[x'_i, x'_{i+2}] \xrightarrow{\sim} [x_i, x_{i+2}]$. Let x_{i+1}, y be the atoms of $[x_i, x_{i+2}]$. By inspection of the dihedral case, we have $T(x_i, x_{i+1}) \prec T(x_{i+1}, x_{i+2})$ if and only if $T(x_i, x_{i+1}) \prec T(x_i, y)$. Since c is lex-minimal, $T(x_i, x_{i+1}) \prec T(x_i, y)$. Hence c is increasing. $\square$

Lemma 5.1.6. *There is exactly one increasing chain in* $[u, v]$.

Proof. We induct on $\ell(v)$. If $\ell(v) - \ell(u) \leq 1$ then we are done. Otherwise, let $s \in S$ be so that $sv < v$. By Lemma 5.1.2, we may freely choose the reflection order we use to compute increasing chains. Pick a reflection order $\prec$ so that $s \preceq t$ for every $t \in T$ (e.g. using an appropriate linear reflection order or picking a maximal chain of biclosed sets beginning with $\{\alpha_s\}$). We break into cases according to whether $su > u$ or $su < u$.

In the first case, if $c = (u = x_0 \lessdot \cdots \lessdot x_k = v)$ is an increasing chain, then we claim some edge of the chain is labeled by s. To see this, note that since $sx_0 > x_0$ and $sx_k < x_k$, there must be some i so that $sx_i > x_i$ and $sx_{i+1} < x_{i+1}$. But by the *Z-property* of Bruhat order (see, e.g., [12]), this implies that the elements sx_i, sx_{i+1} are in $[x_i, x_{i+1}]$, which means that $sx_i = x_{i+1}$. Because s is ordered first by $\prec$, we must furthermore have $i = 0$. And $x_1 \lessdot \cdots \lessdot x_k$ is an increasing chain of $[x_1, x_k]$, so by induction it is the lex-minimal chain of $[x_1, x_k]$. Hence c is the lex-minimal chain, since its first edge has the minimal possible label.

In the second case, let $c = (u = x_0 \lessdot \cdots \lessdot x_k = v)$ be an increasing chain. Then no edge label of c is s, since such a label would have to be labeling $x_0 \lessdot x_1$ and $sx_0 < x_0$, so this is impossible. By similar reasoning to the previous paragraph, we must have $sx_i < x_i$ for all

i. Note that if there is an edge in the Bruhat graph $x_i \to x_{i+1}$ then there is also an edge $sx_i \to sx_{i+1}$. In particular, $sx_i \lessdot sx_{i+1}$ for all i. Hence there is a chain $sc \coloneqq sx_0 \lessdot \cdots \lessdot sx_k$ in $[su, sv]$. Furthermore, sc is an increasing chain for the reflection order[1]$\prec^s$ defined by $t_1 \preceq^s t_2$ if and only if $t_2 = s$ or $st_1s^{-1} \preceq st_2s^{-1}$. Hence we have given an injection from the $\prec$-increasing chains of $[u, v]$ into the $\prec^s$-increasing chains of $[su, sv]$. By induction on $\ell(sv)$, there is exactly one increasing chain of $[su, sv]$, so we are done. □

This completes the proof of Theorem 5.1.1. We now give another proof of the previous lemma that seems to be new. We state it with a hypothesis to avoid circular reasoning, since we will prove the hypothesis in the next section assuming Lemma 5.1.6. A non-circular proof of the hypothesis for finite Weyl groups can be given by applying the simplex method to the vertex figure of a Bruhat interval polytope, using the results from [49].

In the following lemma, by a "cover relation" we mean a cover relation in the poset $[u, v]$.

Lemma 5.1.7. *Assume there exists a reflection order $\prec$ satisfying the following:*

- *For any cover relation $x \lessdot y$ so that there exists a cover relation $x \lessdot y'$ with $T(x, y') \prec T(x, y)$, then there exists a cover relation $y \lessdot z$ so that $T(y, z) \prec T(x, y)$.*

Then there is exactly one increasing chain in $[u, v]$.

Proof. We induct on $\ell(v) - \ell(u)$. We know that the lexicographically minimal chain is increasing, so it is enough to show that there is no other increasing chain. Let $c = (u = x_0 \lessdot \cdots \lessdot x_k = v)$ be a non-lexicographically minimal chain. Then there is some i so that there exists a cover relation $x_i \lessdot y'$ with $T(x_i, y') \prec T(x_i, x_{i+1})$. By our assumption, there is a cover relation $x_{i+1} \prec z$ so that $T(x_{i+1}, z) \prec T(x_i, x_{i+1})$. But $x_{i+1} \lessdot \cdots \lessdot x_k$ is an increasing chain from x_{i+1} to x_k, so by induction it must be the lex-minimal chain in $[x_{i+1}, x_k]$. This contradicts the existence of z. Hence c is not increasing. □

[1]To see that $\prec^s$ is a reflection order, let $\varnothing \lessdot \{\alpha_s\} \lessdot B_2 \lessdot B_3 \lessdot \cdots \lessdot \Phi^+$ be the maximal chain of biclosed sets defining $\prec$, and note that $\prec^s$ is defined by the maximal chain of biclosed sets $\varnothing \lessdot s \cdot B_2 \lessdot s \cdot B_3 \lessdot \cdots \lessdot s \cdot \Phi^+ \lessdot \Phi^+$.

5.2 Bruhat interval polytopes

In this section, we will prove that faces of a Bruhat interval polytope are Bruhat interval polytopes, for an arbitrary Coxeter group. This has been proven for finite Weyl groups using *totally non-negative flag varieties* and the canonical basis in [49]; they also suggest there a proof (due to Knutson) using Frobenius splitting for Richardson varieties. Their results were generalized to finite Coxeter groups in [20]. Related results for the Bruhat interval polytopes of lower Bruhat intervals of affine Weyl groups appear in [11], and one can also extract a proof for all intervals of affine Weyl groups from [10, Corollary 3.3]. The approach taken in this section is motivated by the proof of analogous results for MV polytopes in [9, 10]. I am grateful to Steven Karp for pointing me to the reference [20] and the connection of the material in this section with Coxeter matroids.

Throughout we fix a Coxeter group W with a dualizable root datum $(V, \Pi, V^*, \Pi^\vee)$ so that V is finite-dimensional, with root system Φ. By Proposition 2.1.13, there is an element $\rho \in V$ so that $\langle \Pi^\vee, \rho \rangle > 0$. Such elements are called **dominant regular weights**. Similarly, there is an element $\rho^\vee \in V^*$ so that $\langle \rho^\vee, \Pi \rangle > 0$, which is called a **dominant regular coweight**.

Definition 5.2.1. Let ρ be a dominant regular weight and let $u \leq v \in W$. The **Bruhat interval polytope** of $[u, v]$ is the polytope

$$\mathsf{Pol}_{u,v} \coloneqq \operatorname{conv}\{x\rho \mid x \in [u, v]\}.$$

Bruhat interval polytopes were defined for Weyl groups in [49]; our definition coincides with theirs in the case that W is a Weyl group, Φ is Kac–Moody, and $\langle \alpha^\vee, \rho \rangle = 1$ for all simple coroots $\alpha^\vee \in \Pi^\vee$.

Evidently the vertices of $\mathsf{Pol}_{u,v}$ are a subset of $[u, v]\rho$. Every element $f \in V^*$ determines a face of $\mathsf{Pol}_{u,v}$ as follows: let $M_f \coloneqq \max_{\mu \in \mathsf{Pol}_{u,v}} \langle f, \mu \rangle$. Then the face determined by f is

$$\{\mu \in \mathsf{Pol}_{u,v} \mid \langle f, \mu \rangle = M_f\}.$$

Each face is determined by its vertices, so we write $[u, v]_f := \{x \in [u, v] \mid \langle f, x\rho \rangle = M_f\}$. The first main theorem of this section is the following.

Theorem 5.2.2. *For each $f \in V^*$, the subset $[u, v]_f \subseteq [u, v]$ is a subinterval. That is, there exist $x, y \in [u, v]$ so that $[u, v]_f = [x, y]$.*

We will see in Lemma 5.2.5 that every element of $[u, v]$ arises as a vertex.

The following lemma is useful for constructing total orders: it says that we get a reflection order by ordering the hyperplanes of a Coxeter arrangement according to the order they are hit by an affine line starting from the dominant chamber.

Lemma 5.2.3. *Let Φ be a root system for a finite rank Coxeter group W. Let $f \in V^*$ be such that there is at most one $\alpha \in \Phi$ with $f \in \alpha^\perp$. Then for $g \in V^*$ avoiding a countable set of codimension one subspaces of V^*, the ordering on Φ given by $\alpha \prec \beta$ if $f(\alpha)/g(\alpha) < f(\beta)/g(\beta)$ is a well-defined total order. If furthermore $g(\Pi) > 0$, then $\prec$ is a reflection order.*

Proof. For f/g to define a total order, we need $g(\alpha) \neq 0$ and $f(\alpha)g(\beta) \neq f(\beta)g(\alpha)$ for each $\alpha, \beta \in \Phi$, which is a countable set of conditions. To check that f/g defines a reflection order, let $\gamma = a\alpha + b\beta$ with $\alpha, \beta, \gamma \in \Phi^+$ and $a, b > 0$. Assume $f(\alpha)/g(\alpha) < f(\beta)/g(\beta)$. Then

$$\frac{f(\gamma)}{g(\gamma)} = \frac{ag(\alpha)}{ag(\alpha) + bg(\beta)} \frac{f(\alpha)}{g(\alpha)} + \frac{bg(\beta)}{ag(\alpha) + bg(\beta)} \frac{f(\beta)}{g(\beta)},$$

so $f(\alpha)/g(\alpha) \leq f(\gamma)/g(\gamma) \leq f(\beta)/g(\beta)$. □

The proof of the following is essentially the same as the one given for finite Coxeter groups in [20, Theorem 4.4].

Lemma 5.2.4. *Let $f \in V^*$ be such that $[u, v]_f = \{x, y\}$ for $x \neq y$. Then there is a reflection $t \in T$ so that $tx = y$ and either $x \lessdot y$ or $x \gtrdot y$.*

Proof. Let $f \in V^*$ be so that $[u, v]_f = \{x, y\}$; picking a generic such f, we can assume that there is at most one $\alpha \in \Phi$ with $f \in \alpha^\perp$. Then picking a generic $g \in V^*$ so that $g(\Pi) > 0$, we have that $\frac{f}{g}$ defines a reflection order $\prec$ by Lemma 5.2.3. Let $u \lessdot x_1 \lessdot \cdots \lessdot x_k = x$ be the increasing chain from u to x and let $x = x_k \lessdot \cdots \lessdot x_r = v$ be the increasing chain from x to v. Then writing $T(x_{k-1}, x_k) = t_k$ and $T(x_k, x_{k+1}) = t_{k+1}$, we have that $f(t_k x\rho) = f(x\rho + a\alpha_{t_k})$ and $f(t_{k+1}x\rho) = f(x\rho - b\alpha_{t_{k+1}})$ for some $a, b > 0$. Then $f(x_{k-1}\rho), f(x_{k+1}\rho) \leq f(x_k\rho)$, so $f(\alpha_{t_k}) \leq 0$ and $f(\alpha_{t_{k+1}}) \geq 0$. Hence $t_1 \prec t_2$, so the chain $x_0 \lessdot \cdots \lessdot x_r$ is the increasing chain from u to v. The same argument applies to y, so we conclude that the increasing chain from u to v contains both x and y. Furthermore, by the construction of this chain, if $t_1 \prec \cdots \prec t_k$ are the reflections on this chain, then if $f(\alpha_{t_k}) \leq 0$ and $f(\alpha_{t_{k+1}}) \geq 0$, then the set $\{x_{k-1}, x_k\}$ must be the set $\{x, y\}$. It follows that either $x \lessdot y$ or $y \lessdot x$. $\square$

Given any finite subset $P \subseteq W$, we write $\mathsf{Pol}_P := \{x\rho \mid x \in P\}$. For instance, $\mathsf{Pol}_{u,v} = \mathsf{Pol}_{[u,v]}$. Then we can write $P_f := \{x \in P \mid f(x\rho) = M_f\}$. When W is a finite Coxeter group, every element of $P\rho$ is a vertex because the elements of $P\rho$ lie on the surface of a sphere. When W is infinite, it is no longer true that the elements of $P\rho$ lie on a sphere, but it turns out (assuming dualizability of the root system) that they are all still vertices of Pol_P.

Lemma 5.2.5. *Let $P \subseteq W$ be a finite subset. For every $x \in P$, there exists a functional $f \in V^*$ so that $P_f = \{x\}$. More precisely, if $\rho^\vee$ is a dominant regular coweight, then $P_{x\rho^\vee} = \{x\}$.*

Proof. Let $\rho^\vee$ be a dominant regular coweight. We will show for any $x, y \in W$ that $\langle x\rho^\vee, y\rho\rangle \leq \langle \rho^\vee, \rho\rangle$ with equality if and only if $x = y$. In particular, this implies $P_{x\rho^\vee} = \{x\}$. It is enough to show that $\langle \rho^\vee, w\rho\rangle \leq \langle \rho^\vee, \rho\rangle$ with equality if and only if $w = 1$.

First observe that if $tw > w$ for $t \in T$, then $\langle \alpha_t^\vee, w\rho\rangle = \langle w^{-1}\alpha_t^\vee, \rho\rangle > 0$ so $tw\rho - w\rho = c\alpha_t$ for some $c < 0$. Hence by induction on $\ell(w)$ we have $w\rho - \rho \in \operatorname{cone}(-\Pi)$ for all $w \in W$, and $w\rho = \rho$ if and only if $w = 1$. We deduce $\langle \rho^\vee, w\rho - \rho\rangle \leq 0$ with equality if and only if $w = 1$, which proves the claim. $\square$

In addition to functionals, we can also use biclosed sets to combinatorially select a subset of P. Let $A \subseteq \Phi^+$ be an initial section of a reflection order. (Assuming Conjecture 3.0.5, we may equivalently choose A to be any biclosed set.) Dyer defines the **twisted Bruhat order**

[25] to be the partial order $\leq_A$ on W generated by the relations $x \leq_A tx$ whenever $\ell(x) < \ell(tx)$ and $t \notin A$, or $\ell(x) > \ell(tx)$ and $t \in A$.[2] Define $P_A = \{x \in P \mid x \text{ is } \leq_A\text{-minimal in } P\}$. Note that for each $w \in W$ and dominant regular coweight $\rho^\vee \in V^*$, we have the containment $P_{w\rho^\vee} \subseteq P_{N(w)}$. Furthermore, the twisted Bruhat order $\leq_{N(w)}$ coincides with the Gelfand–Serganova twisted order [14], which puts $x \leq_{N(w)} y$ if and only if $w^{-1}x \leq w^{-1}y$.

We will now prove the second main theorem of this section, a generalization of the Gelfand–Serganova Theorem (also called the "Fundamental Theorem of Coxeter Matroids") [14, Section 6.3] to arbitrary Coxeter groups.

Theorem 5.2.6. *The following are equivalent, for a finite subset P of a Coxeter group W with real root system Φ:*

(a) Every edge of Pol_P is parallel to a root in Φ;

(b) For every dominant regular coweight $\rho^\vee \in V^*$ and every $w \in W$, the set $P_{w\rho^\vee}$ is a singleton;

(c) For every $w \in W$, the set $P_{N(w)}$ is a singleton.

In particular, when these conditions hold, then $P_{N(w)} = P_{w\rho^\vee}$ for every dominant regular coweight $\rho^\vee \in V^*$ and every $w \in W$.

Proof. (a) $\Longrightarrow$ (b): Since every edge of Pol_P is parallel to a real root, the set P_f is a singleton unless f is in $\alpha^\perp$ for some $\alpha \in \Phi$. By definition, a dominant regular coweight is not contained in any $\alpha^\perp$. Hence if $w \in W$ and $\rho^\vee$ is a dominant regular coweight, then $w\rho^\vee$ is not in $\alpha^\perp$ for any $\alpha \in \Phi$, since Φ is W-invariant. It follows that $P_{w\rho^\vee}$ is a singleton.

(b) $\Longrightarrow$ (a): Let $x, y \in W$ be so that $x\rho$ and $y\rho$ are the vertices on an edge of Pol_P. By Lemma 5.2.5, if $\rho^\vee$ is a dominant regular coweight, then $P_{x\rho^\vee} = \{x\}$ and $P_{y\rho^\vee} = \{y\}$. Then the line segment connecting $x\rho^\vee$ to $y\rho^\vee$ passes through a unique point f so that $\{x, y\} \subseteq P_f$. By replacing $x\rho^\vee$ with a generic point p very close to $x\rho^\vee$, we may assume that $P_f = \{x, y\}$.

[2] Dyer's definition and ours differ by inversion. His is "twisted on the right" and ours is "twisted on the left".

Hence f is in the relative interior of the codimension 1 face F of the normal fan of Pol_P defining our edge. As a result, if B_ε is an ε-ball around f, and $B_\varepsilon \cap F$ is contained in $\alpha^\perp$ for some root $\alpha \in \Phi$, then the edge is parallel to α. Since f is in the convex hull of the points $p, y\rho^\vee$, which are in the interior of the Tits cone, f is in the interior of the Tits cone. Since the interior of the Tits cone is open, we may take ε small enough so that B_ε is contained in the interior of the Tits cone. Now, if any point $g \in B_\varepsilon \cap F$ is regular (i.e. not contained in $\alpha^\perp$ for some $\alpha \in \Phi$), then $g = w\rho'^\vee$ for some dominant regular coweight $\rho'^\vee$ and $w \in W$. By (b), this implies $|P_g| = 1$, contradicting the fact that $g \in F$. Hence every point of $B_\varepsilon \cap F$ is in some $\alpha^\perp$; but since B_ε is contained in the interior of the Tits cone, there are only finitely many $\alpha^\perp$ intersecting B_ε. It follows that there is some spanning subset of F contained in a single hyperplane $\alpha^\perp$ with $\alpha \in \Phi$, so the edge is parallel to α.

(c) $\Longrightarrow$ (b): Follows since $P_{w\rho^\vee} \subseteq P_{N(w)}$.

(a+b) $\Longrightarrow$ (c): Conditions (a) and (b) are preserved by the action of W, so we may assume that $w = e$. Hence we wish to show that P has a unique Bruhat minimal element. Note that if $x\rho$ and $y\rho$ are the endpoints of an edge parallel to α_t, then $t(x\rho - y\rho) = ta\alpha_t = -a\alpha_t = y\rho - x\rho$, so $tx = y$. Let $\rho^\vee$ be a dominant regular coweight and set $p = P_{\rho^\vee}$. If $\{x\rho, tx\rho\}$ are the endpoints of an edge in Pol_P, then $x\rho < tx\rho$ if and only if $\langle \rho^\vee, x\rho\rangle > \langle \rho^\vee, tx\rho\rangle$. Hence applying the simplex algorithm to maximize $\rho^\vee$ to any point $x\rho$ with $x \in P$ will result in a Bruhat-decreasing sequence of elements of P that ends at p.

□

Proof of Theorem 5.2.2. Let $f \in V^*$ and set $P \coloneqq [u, v]_f$. Then Pol_P is a face of $\mathsf{Pol}_{u,v}$, so by Lemma 5.2.4, the edges of Pol_P are parallel to roots. By Theorem 5.2.6, there are minimum and maximum elements of P under Bruhat order, say x and y. We claim $P = [x, y]$. Let g be generic so that $g(\Pi) > 0$ and let f' be a generic f in a small neighborhood of f, so that the value of f'/g on edges in P is ≈ 0 and the value on edges leaving P is bounded away from 0. Let $\prec$ be the reflection order defined by f'/g. Let $x \lessdot x_1 \lessdot \cdots \lessdot x_r = y$ be the increasing chain from x to y. If the chain leaves P, then there exist $x_i \lessdot x_{i+1}$ and $x_j \lessdot x_{j+1}$

with $x_i, x_{j+1} \in P$ and $x_{i+1}, x_j \in P$. But the value of f'/g on $T(x_i, x_{i+1})$ is positive while the value of f'/g on $T(x_j, x_{j+1})$ is negative, so this would not be an increasing chain. Hence the chain stays in P. Now any two chains in $[x, y]$ are connected by a sequence of dihedral moves as in the proof of Lemma 5.1.2. Dihedral moves replace two edges spanning a plane with two other edges in the same plane. So applying a dihedral move preserves the property of a chain being contained in P. Hence every chain in $[x, y]$ is contained in P, so $[x, y] = P$. $\square$

5.3 Combinatorial invariance

In this section, we describe joint work with Christian Gaetz reported in [5, 6]. In that paper, the results applied only to S_n. Here we describe also their extension to other groups.

Definition 5.3.1. The $\widetilde{R}$**-polynomial** is the unique family of polynomials $\{\widetilde{R}_{u,v}\}_{u,v \in W}$ so that

- If $u \not\leq v$ then $\widetilde{R}_{u,v} = 0$;
- $\widetilde{R}_{u,u} = 1$;
- If $s \in S$ and $vs < v$, then

$$\widetilde{R}_{u,v} = \begin{cases} q\widetilde{R}_{u,vs} + \widetilde{R}_{us,vs} & \text{if } us > u \\ \widetilde{R}_{us,vs} & \text{if } us < u \end{cases}.$$

The **Kazhdan–Lusztig** R**-polynomial** is

$$R_{u,v}(q) \coloneqq q^{\ell(u,v)/2} \widetilde{R}_{u,v}(q^{1/2} - q^{-1/2}).$$

The **Kazhdan–Lusztig** P**-polynomial** is the unique family of polynomials $\{P_{u,v}\}_{u,v \in W}$ so that

- If $u \not\leq v$ then $P_{u,v} = 0$;

- $P_{u,u} = 1$;

- If $u < v$, then $\deg P_{u,v} < \ell(u,v)/2$ and

$$q^{\ell(u,v)} P_{u,v}(q^{-1}) = \sum_{u \leq x \leq v} R_{u,x}(q) P_{x,v}(q).$$

The following is known as the *combinatorial invariance conjecture*.

Conjecture 5.3.2. *If W is a Coxeter group and $u, v, u', v' \in W$ are so that $[u,v] \cong [u',v']$ as posets, then $P_{u,v} = P_{u',v'}$.*

Because the polynomials $P_{u,v}$ are computed from the polynomials $R_{u,v}$, the combinatorial invariance conjecture is implied by (and, in fact, equivalent to):

Conjecture 5.3.3. *If W is a Coxeter group and $u, v, u', v' \in W$ are so that $[u,v] \cong [u',v']$ as posets, then $\widetilde{R}_{u,v} = \widetilde{R}_{u',v'}$.*

In joint work with Christian Gaetz [5], we proved many cases of this conjecture for $W = S_n$. We use *hypercube decompositions*, introduced in [13]. Let Γ denote the Bruhat graph of W. We write $x \to y$ if there is an edge of the Bruhat graph from x to y. In the following definition, we use the fact that when $x \to y \to z$ is a path of length 2 in Γ, then there are either 2 or 4 paths of length 2 from x to z. Furthermore, if there are 4 paths of length 2, and the middle vertices of the paths are $\{y_1, y_2, y_3, y_4\}$, then there is some indexing so that $y_1, y_2 \leq y_3, y_4$.

Definition 5.3.4. Let $\mathcal{H}_n$ be the **hypercube graph** on n elements. This is the directed graph which is the Hasse diagram of the Boolean lattice with n atoms. An n-**hypercube** is a subgraph of Γ isomorphic to $\mathcal{H}_n$. A 2-hypercube is called a **diamond**. This is a subgraph of Γ having (distinct) vertices $\{x_1, x_2, x_3, x_4\}$ with directed edges $x_1 \to x_2 \to x_4$ and $x_1 \to x_3 \to x_4$. Let $y_1, y_2, \ldots, y_{2r}$ be the set of vertices in Γ such that $x_1 \to y_i \to x_4$, ordered so that $y_i \leq y_j$ implies $i \leq j$. Then the diamond $\{x_1, x_2, x_3, x_4\}$ is called **special** if there is an integer i such that $\{x_2, x_3\} = \{y_{2i+1}, y_{2i+2}\}$.

Given a sequence of two edges $x_1 \to x_2 \to x_4$ in Γ, their **diamond flip** is the unique pair of edges $x_1 \to x_3 \to x_4$ such that $\{x_1, x_2, x_3, x_4\}$ is a special diamond.

Given sets $X \subset Y \subset S_n$, we say X is **diamond-closed** in Y if whenever X contains three vertices of a diamond contained in Y, it also contains the fourth.

Definition 5.3.5. Let $I \subset [u, v]$ be an order ideal and let $x \in I$. Define

$$\mathcal{Y}_x = \{y \in [u, v] \setminus I \mid x \to y\}$$

and

$$\mathcal{A}_x = \{Y \subset \mathcal{Y}_x \mid Y \text{ is an antichain in Bruhat order}\}.$$

The collection $\mathcal{A}_x$ is naturally a poset under inclusion order. We also view it as a directed graph where $Y_1 \to Y_2$ if $Y_1 \subset Y_2$ and $|Y_2 \setminus Y_1| = 1$. Call the elements of $\mathcal{A}_x$ **antichains over** x (relative to I).

A **strong hypercube cluster** at x (relative to I) consists of a function $\theta_x : \mathcal{A}_x \to [u, v]$, called the **hypercube map**, satisfying the following four properties:

(HC1) $\theta_x(\varnothing) = x$.

(HC2) If $y \in \mathcal{Y}_x$, then $\theta_x(\{y\}) = y$.

(HC3) If $Y_1 \to Y_2$, then $\theta_x(Y_1) \to \theta_x(Y_2)$.

(HC4) If $|Y| = |Y'| = |Y \cap Y'| + 1$ and we have a special diamond in $\Gamma(u, v)$ of the form

$$\begin{tikzcd} & w & \\ \theta_x(Y) \arrow[ur] & & \theta_x(Y'), \arrow[ul] \\ & \theta_x(Y \cap Y') \arrow[ul] \arrow[ur] & \end{tikzcd}$$

then $Y \cup Y'$ is an antichain and $\theta_x(Y \cup Y') = w$.

If there is a hypercube cluster at x, then let $\mathcal{H}_x^y = \{Y \in \mathcal{A}_x \mid \theta_x(Y) = y\}$ denote the collection of hypercubes with bottom vertex x and top vertex y.

We also define the *relative $\widetilde{R}$-polynomial* of an order ideal $I \subseteq [u, v]$. This is a generalization of the relative $\widetilde{R}$-polynomial introduced in [33].

Definition 5.3.6. Let $I \subseteq [u, v]$ be a nonempty order ideal. The **relative $\widetilde{R}$-polynomial** is

$$\widetilde{R}_{u,v,I} := \sum_{x \in I} \widetilde{R}_{u,x}(-q)\widetilde{R}_{x,v}(q).$$

We have the following, which lets us recover $\widetilde{R}$ from the relative $\widetilde{R}$-polynomial and the $\widetilde{R}$-polynomial of strictly smaller intervals.

Lemma 5.3.7. *Let $I \subseteq [u, v]$ be a nonempty order ideal. Then*

$$\widetilde{R}_{u,v}(q) = \sum_{x \in I} \widetilde{R}_{u,x}(q)\widetilde{R}_{x,v}(q)$$

and

$$\sum_{y \in [u,v] \setminus I} \widetilde{R}_{u,y}(-q)\widetilde{R}_{y,v}(q) = 0.$$

We will be interested in a certain family of intervals which includes intervals of the form $[e, v]$. Recall $T(u, a)$ denotes the reflection labeling a cover relation.

Definition 5.3.8. An interval $[u, v]$ in a Coxeter group W is called **simple** if $\{\alpha_t \mid t = T(u, a),\ u \lessdot a \leq v\}$ is a linearly independent set of vectors.

One of the main results in [5], stated in the updated framework from [6], is the following.

Theorem 5.3.9. *Let W be a Coxeter group and assume $[u, v]$ is a simple interval. If I is a principal diamond-closed order ideal in $[u, v]$ and there is a hypercube cluster at u, then for any $y \in [u, v]$*

$$\widetilde{R}_{u,y,I} = \sum_{Y \in \mathcal{H}_u^y} q^{|Y|}.$$

In [5], we derived the following corollary using the existence of *standard hypercube decompositions* (introduced in [13]) in S_n. It encompasses the previous broadest case of combinatorial invariance known, the case where $u = u' = e$.

Corollary 5.3.10. *If $W = S_n$ and u, v, u', v' are such that $[u, v] \cong [u', v']$ and there exists a simple interval isomorphic to both intervals, then $\widetilde{R}_{u,v} = \widetilde{R}_{u',v'}$.*

The main result of [6] applies the relative $\widetilde{R}$-polynomial to Kazhdan–Lusztig P polynomials, resolving a conjecture of DeepMind and Williamson [13] in the case where $u = e$.

Theorem 5.3.11. *Let $v \in S_n$ and let I be a hypercube decomposition of $[e, v]$. Then*

$$\sum_{y,y' \in [e,v] \setminus I} \sum_{Y \in \mathcal{H}_e^y} (-1)^{|Y|} q^{\ell(y) - |\Lambda(Y)|} (q-1)^{|Y|} R_{y,y'} P_{y',v} = 0.$$

Here $\Lambda(Y)$ denotes the set $\{y \in \mathcal{Y}_u \mid \exists y' \in Y,\ y \leq y'\}$.

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