Problem Set 1

1. Warm up: compute the one-line notation of the permutation $(2,4)(2,3)(1,4)(1,3)(1,2)$ in S_4.

2. Show that the simple transpositions generate S_n.

3. Show that the reflections in S_n are the transpositions.

4. Show that every non-identity element of a Coxeter group has a descent. If s is a descent of w, show that $\ell(ws) = \ell(w) - 1$.

5. In the following problems, feel free to use the fact that, for $a < b$, having $(a,b) \in \text{inv}(\pi)$ means that b is to the left of a in the one-line notation of π (equivalently, $\pi^{-1}(b) < \pi^{-1}(a)$).
   (a) Compute inv(3412). What is $\ell(3412)$?
   (b) Prove that there is a unique element in S_n with maximal length. What is its length? This element is often called w_0.
   (c) Prove that there is a unique reflection in S_n with maximal length among reflections. What is its length?

6. The dihedral group $I_2(m)$ is generated by an order-2 reflection s_1 and an order-m rotation R. Define $s_2 = s_1 R$. Let $S = \{s_1, s_2\}$ and show that $(I_2(m), S)$ is a Coxeter system. Show that the Coxeter diagram is $1 \overset{m}{\text{———}} 2$.

7. Let the group B_n be defined by

$$B_n := \{\pi \mid \pi \text{ is a permutation of } \{-n, -n+1, \ldots, n-1, n\} \text{ such that } \forall i, \pi(-i) = -\pi(i)\}.$$

   Define the elements $s_1 = (1,2)(-1,-2)$, $s_2 = (2,3)(-2,-3)$, $\ldots$, $s_{n-1} = (n-1,n)(-n+1,-n)$ and the element $s_0^B = (-1,1)$. Let $S = \{s_0^B, \ldots, s_{n-1}\}$. Then (B_n, S) is a Coxeter system (optional exercise: prove it).
   (a) What is its Coxeter diagram?
   (b) How many elements are in B_n?
   (c) How many reflections are in B_n?
   The group B_n is also sometimes called C_n.

8. Define a partial order on partitions of length k so that $\lambda \leq \mu$ if and only if $\lambda_i \leq \mu_i$ for all i.
   (a) Let X be the set of partitions of length 2 which are $\leq (2,2)$. (There are 6 elements of X.) Draw the Hasse diagram for X.
   (b) Can you describe the cover relations in this partial order?

9. Verify that the Hasse diagrams for the weak order and Bruhat order on S_3 match the ones from the lecture.

10. Prove that if two distinct elements of a Coxeter group have the same length, then they are incomparable in Bruhat order.

11. There are several important properties of Coxeter groups that are all equivalent to the definition. Try proving a few from our definition:
    (a) Strong Exchange Property: If $t \in \operatorname{inv}(w)$ and $s_{i_1} \cdots s_{i_k}$ is a reduced word for w, then there exists a j so that $tw = s_{i_1} \cdots \widehat{s}_{i_j} \cdots s_{i_k}$. (The hat notation means to omit s_{i_j} from the word.)
    (b) Exchange Property: If s is a descent of w and $s_{i_1} \cdots s_{i_k}$ is a reduced word for w, then there exists a j so that $ws = s_{i_1} \cdots \widehat{s}_{i_j} \cdots s_{i_k}$.
    (c) Deletion Property: If $s_{i_1} \cdots s_{i_k}$ is a non-reduced word for w, then there exist j and j' so that $w = s_{i_1} \cdots \widehat{s}_{i_j} \cdots \widehat{s}_{i_{j'}} \cdots s_{i_k}$.

12. Let the group D_n be defined by

$$D_n := \{\pi \in B_n \mid \text{The size } |\{i \in [n] \mid \pi(i) < 0\}| \text{ is even}\}.$$

Define the elements $s_1, \ldots, s_{n-1}$ as above, and define $s_0^D = (-1,2)(-2,1)$. Let $S = \{s_0^D, \ldots, s_{n-1}\}$. Then (D_n, S) is a Coxeter system (optional exercise: prove it). What is its Coxeter diagram?

13. Let W be a Coxeter group and let $w \in W$ and $t \in T$. Prove that $\ell(tw) - \ell(w)$ is odd.

14. Let W be a Coxeter group and let $t \in T$ be a reflection. Prove that t has a palindromic reduced word.

15. Draw the Hasse diagrams for weak order and Bruhat order on $I_2(4)$.

16. Prove that if (x, y) is a cover relation in weak order, then $\ell(y) = \ell(x) + 1$. Do the same for Bruhat order.

17. Prove that in the weak order on W, the number of elements which are covered by x is at most the rank of W. Is the same true for Bruhat order?

18. Prove that B_3 is isomorphic to the group generated by the reflective symmetries of a cube.